A Jacquet-Langlands relation between mod p Hilbert and quaternionic modular forms

Fred Diamond*

King's College London

25 July 2018, Rio de Janeiro
ICM satellite conference on Automorphic Forms, Galois Representations and L-functions

* joint with Payman Kassaei and Shu Sasaki

The case of modular curves

A relation between mod p modular forms

- of weight 2, level Np ($p \nmid N$)
- and of weight $k \in [2, p+1]$ and level N:

Theorem (Serre)

Suppose $p \nmid N$, $N > 3$. Let $\mathcal{K}$ be the dualizing sheaf on $X_1(Np)_{\mathbb{F}_p}$, and define $\chi_m : \mathbb{F}_p \to \mathbb{F}_p$ by $x \mapsto x^m$ for $m = 1, \cdots, p-1$. Then there is a Hecke-equivariant exact sequence:

$$0 \to H^0(X_1(N)_{\mathbb{F}_p}, \delta^m \omega^{p+1-m}) \to H^0(X_1(Np)_{\mathbb{F}_p}, \mathcal{K}(\infty))^{\chi_m} \\ \to H^0(X_1(N)_{\mathbb{F}_p}, \omega^{m+2}) \to 0,$$

where δ is a trivial bundle twisting the action of T_q by q.

Motivating question

The relation reflects the exact sequence
on étale cohomology arising from

$$0 \to \det{}^m \mathrm{Sym}^{p-1-m}\mathbb{F}_p^2 \to \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_p)}(\chi_m \otimes 1) \to \mathrm{Sym}^m \mathbb{F}_p^2 \to 0$$

Question: How does this generalize to the
Hilbert modular setting?

The mod p geometry of $X_1(Np)$ is more complicated.
So is the structure of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_{p^r})}\chi$.
(Typically 2^r Jordan-Holder constituents.)

The answer involves a mod p Jacquet-Langlands relation.

Hilbert modular varieties

F totally real field, $d = [F : \mathbb{Q}]$, p unramified in F,
Fix $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \subset \mathbb{C}$, so identifications:

$$\Sigma = \{F \to \mathbb{R}\} \leftrightarrow \{F \to \overline{\mathbb{Q}}_p\} \leftrightarrow \{O_F/p \to \overline{\mathbb{F}}_p\}.$$

Fix $U \subset \mathrm{GL}_2(A_F^\infty)$, sufficiently small tame level.

Let X be the Hilbert modular variety of level U over $\overline{\mathbb{F}}_p$.
So X is (the quotient by $O_{F,+}^\times$ of) a scheme representing Hilbert-Blumenthal abelian varieties with additional structure.

HMV's of level p

Let $X_0(p)$ (resp. $X_1(p)$) be the HMV over $\overline{\mathbb{F}}_p$ of level $U \cap U_0(p)$ (resp. $U \cap U_1(p)$) defined by Pappas.
So $X_0(p)$ represents suitable degree-p^d isogenies $f : A \to A'$ of HBAV's (mod $O_{F,+}^\times$),
and $X_1(p)$ is a closed subscheme of $\ker(f)$.

- X is smooth of dimension d,
- $X_0(p)$ is a complete intersection,
- $X_1(p)$ is finite flat over $X_0(p)$, hence Cohen-Macaulay.

Main object of interest:

Let $\mathcal{K}_1$ be the dualizing sheaf on $X_1(p)$, and consider

$$H^0(X_1(p), \mathcal{K}_1) = \oplus_\chi H^0(X_1(p), \mathcal{K}_1)^\chi$$

where χ runs over characters $(O_F/p)^\times \to \overline{\mathbb{F}}_p^\times$.

Since $\pi : X_1(p) \to X_0(p)$ is finite flat,

$$\pi_*\mathcal{K}_1 = \mathcal{H}om_{\mathcal{O}_{X_0(p)}}(\pi_*\mathcal{O}_{X_1(p)}, \mathcal{K}_0),$$

where $\mathcal{K}_0$ is the dualizing sheaf on $X_0(p)$.
So $\pi_*\mathcal{K}_1 = \oplus_\chi \mathcal{L}_\chi \otimes \mathcal{K}_0$ where $\mathcal{L}_\chi^{-1} = (\pi_*\mathcal{O}_{X_1(p)})^{\chi^{-1}}$,
and we're interested in

$$H^0(X_0(p), \mathcal{L}_\chi \otimes \mathcal{K}_0)$$

(weight 2 modular forms of level $U_1(p)$, character χ).

Components of $X_0(p)$

$X_0(p)$ has 2^d types of irreducible components, indexed by subsets $\eta \subset \Sigma$:

$$X_0(p) = \bigcup_{\eta \subset \Sigma} X_\eta,$$

where the X_η are the top Goren-Kassaei strata, so X_η is smooth of dimension d, defined by:

- $\mathrm{Lie}(f^\vee)_\beta = 0$ for $\beta \in \eta$
- $\mathrm{Lie}(f)_{\phi^{-1}\beta} = 0$ for $\beta \notin \eta$

where f is the universal isogeny over $X_0(p)$.

Let $i_\eta : X_\eta \to X_0(p)$.

Lemma

There is a filtration $0 \subset \mathrm{Fil}^d \mathcal{K}_0 \subset \cdots \subset \mathrm{Fil}^0 \mathcal{K}_0 = \mathcal{K}_0$ *such that* $\mathrm{gr}^m \mathcal{K}_0 = \bigoplus_{|\eta|=m} \mathcal{F}_\eta$ *and*

$$\mathcal{F}_\eta = i_{\eta *} \mathcal{K}_\eta (\sum_{\beta \notin \eta} Z_{\eta,\beta}),$$

where $\mathcal{K}_\eta$ *is dualizing on* X_η *and* $Z_{\eta,\beta}$ *is the divisor defined by the intersection* $X_\eta \cap X_{\eta \cup \{\beta\}}$.

Key point: for each β, $Y_\beta = \cup_{\beta \in \eta} X_\eta$ and $Y'_\beta = \cup_{\beta \notin \eta} X_\eta$ are complete intersections, giving a corresponding short exact sequence. Combine these to get the filtration.

Upshot

Now we have a filtration on $(\pi_* \mathcal{K}_1)^\chi$
whose graded pieces are direct sums of line bundles supported on the X_η:

$$\mathcal{G}_\eta := i_\eta^* \mathcal{L}_\chi \otimes \mathcal{K}_\eta(\sum_{\beta \notin \eta} Z_{\eta,\beta}),$$

To understand these, we first prove
(inspired by Pappas, Helm, Tian-Xiao)
that the X_η are isomorphic to products of $\mathbb{P}^1$'s
over quaternionic Shimura varieties.

Quaternionic Shimura varieties

For each η, define $\Sigma_\eta \subset \Sigma = \{F \to \mathbb{R}\}$ corresponding to

$$\{\beta \in \eta \,|\, \phi \circ \beta \notin \eta\} \cup \{\beta \notin \eta \,|\, \phi \circ \beta \in \eta\}.$$

(Note that $|\Sigma_\eta|$ is even.)
Let D_η be the quaternion algebra over F ramified at exactly Σ_η (so unramified at all finite places).
Choose/identify $(D_\eta \otimes \mathbb{A}^\infty)^\times = \mathrm{GL}_2(\mathbb{A}_F^\infty)$.
Let X^{D_η} be the (reduction of the canonical model of the) quaternionic Shimura variety of level (corresponding to) U.
So X^{D_η} is smooth of dimension $d - |\Sigma_\eta|$.

Theorem* (DKS)

X_η is isomorphic to the fibre product over X^{Σ_η} of the $\mathbb{P}(\mathcal{V}_\beta)$ for $\beta \in \Sigma_\eta$, where $\mathcal{V}_\beta$ is a rank two automorphic bundle on X^{Σ_η}.
Moreover the isomorphisms (for varying U) are Hecke-equivariant.

$*$ - proved analogous result for the corresponding unitary Shimura varieties, still checking details of transfer.

Remarks

- The map (corresponding to) $\pi_\eta : X_\eta \to X^{\Sigma_\eta}$
  is defined by $(A \to A') \mapsto B$,
  where $A \to B \to A'$ is determined by η.
- $\mathcal{V}_\beta$ is defined by $H^1_{\mathrm{dR}}(B/S)_\beta$.
- The X_η were known to be products of $\mathbb{P}^1$'s
  over strata in X (up to Frobenius factors),
  which in turn were known to be products of $\mathbb{P}^1$'s
  over quaternionic Shimura varieties,
  but the composite doesn't give the above.

The graded pieces

Now we compute the factors $\mathcal{L}_\chi$, $\mathcal{K}_\eta$, $Z_{\eta,\beta}$ of the constituents of $(\pi_*\mathcal{K}_1)^\chi$:

- Can write each $\mathcal{L}_\chi$ as a product of powers of $\mathcal{L}_\beta$ (associated to corresponding fundamental characters).
- $Z_{\eta,\beta} = \mathcal{L}_\beta^{-1}\mathcal{L}_{\phi\circ\beta}^p$ (like partial Hasse invariants).
- Write each $\mathcal{L}_\beta$ in terms of tautological or automorphic bundles (according to whether $\beta \in \Sigma_\eta$), with signs determined by whether $\beta \in \eta$;
- $\mathcal{K}_\eta$ is the product of the $\mathcal{O}(-2)_\beta$ for $\beta \in \Sigma_\eta$ and $\pi_\eta^*\omega_\beta^2$ for $\beta \notin \eta$ (up to $\pi_\eta^*\delta_\beta^{\pm 1}$).

The main result

Putting all this together, get $\mathcal{G}_\eta^{\Sigma_\eta} := \pi_{\eta,*}\mathcal{G}_\eta$
is precisely the automorphic bundle on X^{B_η}
whose weight matches the corresponding summand of

$$\mathrm{Ind}_B^{\mathrm{GL}_2(O_F/p)}(\chi \otimes 1).$$

Moreover the $R^i\pi_{\eta,*}\mathcal{G}_\eta = 0$ for $i > 0$.

Theorem*

There is a Hecke-equivariant spectral sequence:

$$E_1^{m,n} = \bigoplus_{|\eta|=m} H^{m+n}(X^{\Sigma_\eta}, \mathcal{G}_\eta^{\Sigma_\eta}) \Rightarrow H^{m+n}(X_1(p), \mathcal{K}_1)^\chi.$$

Corollary*

There is a filtration on $H^0(X_1(p), \mathcal{K}_1)$ *such that*

$$\mathrm{gr}^m \hookrightarrow \oplus_{|\eta|=m} H^0(X^{\Sigma_\eta}, \mathcal{G}_\eta^{\Sigma_\eta})$$

We don't know if these are surjective.
(Can construct examples with $H^1(X^{\Sigma_\eta}, \mathcal{G}_\eta^{\Sigma_\eta}) \neq 0$.)
Can at least hope the cokernels are Eisenstein...