$\mu_{\text{Gal}} \le \mu_{\text{Aut}}$

Shu Sasaki

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VERY PRELIMINARY DRAFT!

1 Introduction

These are lecture notes based on the one and half hour long lecture I gave at the workshop (17/12/2012-18/12/2012) about the Breuil-Mezard conjecture in Luxembourg.

The organisers of the workshop had asked me to explain Kisin's paper [4], in particular, 1.6.7 through to 1.7.16 of the paper, and the lecture inevitably followed the paper faithfully, given that my task was to explain Kisin's proofs.

Let p be a rational *odd* prime and let E be a finite extension of $\mathbf{Q}_p$. The p-adic local Langlands correspondence, as formulated by Breuil, 'sends' a two-dimensional continuous E-linear representation V of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ to an admissible, unitary Banach space representation $B(V)$ of $GL_2(\mathbf{Q}_p)$ defined over E, which is in compatible with the classical local Langlands correspondence and with the mod p local Langlands correspondnce for these groups.

Colemz's construction of the p-adic local Langlands correspondence is via Fontaine's theory of (φ,Γ)-modules. In particular, he proves an equivalence of categories by which one associates to V as above a (φ,Γ)-module $D(V)$, and, when V is irreducible, construct $B(V)$ from $D(V)$; he also makes the observation that one can perform $V \mapsto D(V) \mapsto B(V)$ 'integrally', i.e., given a finite type $\mathcal{O}_E$ module $T \subset V$, one can build $D(T) \subset D(V)$ and then $B(T) \subset B(V)$ compatibly; and furthermore that one can conversely recover $D(T)$ from $B(T)$. The last observation, dubbed $\mathbf{V}$, seems to be at the heart of Kisin's proof of the Breuil-Mézard and the Fontaine-Mazur conjectures, not least because the integrality allows him to 'count' representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and representations of $GL_2(\mathbf{Q}_p)$ by deformations (see [3]).

2 Kisin's paper

We shall do our best following Kisin's notation (with minor alterations); local class field theory is normalised so that uniformisers correspond to geometric Frobenii. Let

$G_{\mathbf{Q}_p}$: the decomposition group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$;

$I_{\mathbf{Q}_p}$: the inertia subgroup of $G_{\mathbf{Q}_p}$;

E: a 'sufficiently large' finite extension of $\mathbf{Q}_p$ with ring $\mathcal{O}$ of integers, a uniformiser π, and residue field $\mathbf{F}$;

χ_{cyclo}: the p-adic cyclotomic character;

ω_{cyclo}: the mod p cyclotomic character;

$\overline{\rho}$: a two-dimensional representation over $\mathbf{F}$ of $G_{\mathbf{Q}_p}$;

k: an integer ≥ 2;

τ: a representation $I_{\mathbf{Q}_p} \to GL_2(E)$ with open kernel of Galois type (i.e. it extends to a representation of the Weil group of $\mathbf{Q}_p$);

$\sigma(\tau)$: the unique finite-dimensional irreducible representation of $GL_2(\mathbf{Z}_p)$ over $\overline{\mathbf{Q}}_p$ associated by Hanniart's inertial local Langlands σ in the appendix to the Breuil-Mézard paper ;

$\sigma_{\text{cr}}(\tau)$: $\sigma(\tau)$ but additionally characterised by $N=0$;

$\sigma(k,\tau) = \sigma(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$;

$\sigma_{\text{cr}}(k,\tau) = \sigma_{\text{cr}}(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$;

$L_{k,\tau}$: a $GL_2(\mathbf{Z}_p)$-stable $\mathcal{O}$-lattice in $\sigma(k,\tau)$;

$L_{k,\tau,\mathrm{cr}}$: a $GL_2(\mathbf{Z}_p)$-stable $\mathcal{O}$-lattice in $\sigma_{\mathrm{cr}}(k,\tau)$;
$\overline{L}_{k,\tau}$: $L_{k,\tau}/\pi$
$\overline{L}_{k,\tau,\mathrm{cr}}$:$L_{k,\tau,\mathrm{cr}}/\pi$
ψ: a character $\mathbf{Q}_p^\times \to \mathcal{O}^\times$ such that its restriction to $\mathbf{Z}_p^\times$ equals $\chi_{\mathrm{cyclo}}^{k-2}(\det\tau)|_{\mathbf{Z}_p^\times}$, which is the central character of $\sigma(k,\tau)$;
$R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$: the universal pseudo-deformation ring of $\mathrm{tr}\,\overline{\rho}$, thought of as a pseudo-representation of dimension two;
$R(\overline{\rho})$: the universal deformation $W(\mathbf{F})$-algebra of $\overline{\rho}$;
$R^{\square}(\overline{\rho})$: the universal framed deformation $W(\mathbf{F})$-algebra of $\overline{\rho}$;
$R^{\square,\psi}(\overline{\rho})$: the universal framed deformation $W(\mathbf{F})$-algebra of $\overline{\rho}$ with determinant $\psi\chi_{\mathrm{cyclo}}$.

With k,τ,ψ as above fixed (and we will), we say that a two-dimensional representation V with a continuous action of $G_{\mathbf{Q}_p}$ is *of type*

$$\mathcal{D} = (k,\tau,\psi)$$

if V is potentially semi-stable of type τ with Hodge-Tate weights $0, k-1$ and determinant $\psi\chi_{\mathrm{cyclo}}$. By a two-dimensional pseudo-representation $G_{\mathbf{Q}_p}$ of type $\mathcal{D}$, we shall mean that it is the trace of a two-dimensional representation V of type $\mathcal{D}$.

Let $R^{\square,\psi}(k,\tau,\overline{\rho})$, $R^{\psi}(k,\tau,\overline{\rho})$ be quotients of $R^{\square}(\overline{\rho})\otimes_{W(\mathbf{F})}\mathcal{O}$, $R(\overline{\rho})\otimes_{W(\mathbf{F})}\mathcal{O}$ respectively as defined in Proposition 1.1.1.

Suppose A is a noetherian local ring with maximal ideal $\mathfrak{m}$. For a finite A-module M, let $e(M,A)$ (or $e(A)$ if $M = A$) denote the Hilbert-Samuel multiplicity in M of irreducible representations over $A/\mathfrak{m}$. Suppose furthermore that M comes equipped with an action of a group G. If Σ is a set of irreducible representations of G on finite-dimensional $A/\mathfrak{m}$-vector spaces. Let $e_\Sigma(M,A)$ denote the 'Hilbert-Samuel multiplicity' in M of representations isomorphic to representations in Σ (see section 1.3 of [4]).

2.1 So far

Let $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$ be the universal pseudo-deformation ring of $\mathrm{tr}\,\overline{\rho}$; as p is odd, the ring pro-represents the functor of continuous pseudo-deformations of $\mathrm{tr}\,\overline{\rho}$ (see Lemma 1.4.2). One of the reasons Kisin works with 'pseudo-deformations' of $\mathrm{tr}\,\overline{\rho}$ rather than 'deformations' of $\overline{\rho}$ seems to be that one has to deal with cases where $\overline{\rho}$ is reducible split, and pseudo-deformation theory seems to work better. Furthermore, we shall only deal with two-dimensional p-adic ($p > 2$) representations, and knowing traces is enough to know their characteristic polynomials.

We shall work with subspaces of $\mathrm{Spec}\,R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$; for example, the subspace of pseudo-deformations of dimension 2 which are traces of representations, or indeed 'of type $\mathcal{D}$' may be demanded; in that case, if a point $t : \mathrm{Spec}\,\mathcal{O}_t \to \mathrm{Spec}\,R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$, defined over the integers $\mathcal{O}_t$ of an extension E_t of E, corresponds to a two-dimensional representation V_t defined over $\mathcal{O}_t$, it is of type $\mathcal{D}$ and the Colmez functor (see Theorem 2.1.1 in [3]) allows one to work on the $GL_2(\mathbf{Q}_p)$-side, i.e., an $\mathcal{O}_t$-admissible lattice Π_t with a central character such that $\mathbf{V}(\Pi_t) \subset V_t$ and $\mathbf{V}(\Pi_t)\otimes\mathbf{Q}_p \simeq V_t$, and there is a map $\text{c-Ind}_{KZ}^{G} L_{k,\tau} \to \Pi_t$ of G-representations. With a view to applying Taylor-Wiles and Wiles' approach to modular lifting theorems, this may be thought of as a local manifestation of the existence of a modular lifting; in proving a 'local $R = T$' in which to count multiplicities, it is hence useful to have a local analogue of 'Hecke modules' and 'patching', and this is somehow what Kisin seems to construct: 'points' of $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$ are defined over any local Artinian rings with residue field $\mathbf{F}$, and, in order to align where they are defined, Kisin constructs an admissible $\mathcal{O}_E$-lattice $\Pi(t)$, the image of $\text{c-Ind}_{KZ}^{G} L_{k,\tau}$ by the aforementioned map, in Π_t. One can do this for a finite set U of deformations t of type $\mathcal{D}$, and let $\Pi(U)$ denote the corresponding admissible $\mathcal{O}$-lattice with G-action and $V(U) = \mathbf{V}(U)$. If U is a countable set of deformations of type $\mathcal{D}$, define $\Pi(U)$ to be the inverse limit of $\Pi(U_{\mathrm{fin}})$ for finite subsets U_{fin} of U, and also $V(U)$ to be the inverse limit of $\mathbf{V}(U_{\mathrm{fin}})$.

Lemma 1 *Let U be a countable set of pseudo-deformations of $\mathrm{tr}\,\overline{\rho}$ of type $\mathcal{D}$. Then $V(U)$ is a finite $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$-module of dimension ≤ 2. Let*

$$R_U^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$$

denote the image of $R^{\mathrm{ps}}(\operatorname{tr}\overline{\rho})$ in $\operatorname{End}(V(U))$. Then $R_U^{\mathrm{ps}}(\operatorname{tr}\overline{\rho})$ is a flat $\mathcal{O}$-algebra of relative dimension ≤ 1.

Proof. This is Lemma 1.6.6. As explained in Lemma 1.6.3, $V(U)$ is a $R_U^{\mathrm{ps}}(\operatorname{tr}\overline{\rho})$-module by the 'specialisation to U' of the universal pseudo-deformation over $R_U^{\mathrm{ps}}(\operatorname{tr}\overline{\rho})$. □

For an integer $0 \leq r \leq p-1$, $\lambda \in \mathbf{F}$, and a character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$, let

$$\pi(r, \lambda, \chi)$$

denote the Barthel-Livné representation of $GL_2(\mathbf{Q}_p)$ over $\mathbf{F}$ as defined in 1.2 [4].

Suppose that Π is an admissible $\mathcal{O}$-lattice (1.2.5), i.e., a representation of $GL_2(\mathbf{Q}_p)$ on an $\mathcal{O}$-module, which is p-adically complete and separated and which admits a central character $\mathbf{Q}_p^\times \to \mathcal{O}^\times$. For every n, let $\Pi_n = \Pi \otimes_{\mathbf{Z}} \mathbf{Z}/p^n$; in which case Π is the inverse limit of the Π_n, and every Π_n is of finite length and admissible (1.2.3) and therefore its Jordan-Hölder factor is either one-dimensional or an infinite-dimensional subquotient of a Barthel-Livné representation. In 1.6.4, Kisin explains how to 'reverse' the construction: given a representation ρ ('Π_1') of $GL_2(\mathbf{Q}_p)$ over $\mathbf{F}$, and a finite collection P of Barthel-Livné representations, all with a central character ψ (fixed), let

$$\rho_P$$

denote the inverse limit ('Π') of finite length quotients ('Π_n') of ρ all of whose Jordan-Holder factors are isomorphic to subquotients of representations in P; one may think of it as the 'completion of ρ at P'. A useful thing about the construction $(\rho, P) \mapsto \rho_P$ is that, if ρ and P are both 'explicit', and in particular if ρ is compactly induced from an irreducible KZ-representation on a finite-dimensional $\mathbf{F}$-vector space, one can make the admissible $\mathcal{O}$-lattice ρ_P explicit. This is the content of Lemma 1.6.5, and it is repeatedly used to great effect in proving Lemma 1.6.6 and Lemma 1.6.8 in which, with a view to understanding $R(U)/\pi$, the structure of the $R(U)/\pi$-module $V(U)/\pi$ is studied. The underlying idea seems to be as follows: for brevity, suppose that U consists of exactly one pseudo-deformation $t = \operatorname{tr} V_t$ of $\operatorname{tr}\overline{\rho}$ of type $\mathcal{D}$; it is hence $\Pi(t)/\pi$ that one needs to understand. Recall that $\Pi(t)$ is an admissible $\mathcal{O}$-lattice defined as the closure of the image of $\text{c-Ind}_{KZ}^G L_{k,\tau}$ in the admissible $\mathcal{O}_t$-lattice Π_t such that $\mathbf{V}(\Pi_t) = V_t$, and one may loosely think of $\Pi(t)$ as the 'completion' of $\text{c-Ind}_{KZ}^G \overline{L}_{k,\tau}$ in Π_t. But 'completion' with respect to what? It should be Jordan-Hölder factors of Π_t/π which are Barthel-Livné representations with a central character! It is in view of Lemma 1.6.5 then that one considers a KZ-module filtration of $\overline{L}_{k,\tau}$ so that, on each KZ-irreducible quotient, its compact induction has an explicit form of 'completion' and $\Pi(t)/\pi$ admits a more amenable description.

2.2 Goal

Our goal is to prove the inequality:

$$e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}) \stackrel{\mathrm{def}}{=} \sum_{n\in\{0,1,\ldots,p-1\}, m\in\{0,1,\ldots,p-2\}} a_{n,m}\mu_{n,m}(\overline{\rho}),$$

where $a_{n,m}$ is the multiplicity of $\sigma_{n,m} = \operatorname{Sym}^n \det^m \mathbf{F}^2$ in the semi-simplification of $\overline{L}_{k,\tau}$, and where $\mu_{n,m}(\overline{\rho}) \in \{0,1,2\}$ is an integer explicitly defined (except the $n = p-2$ semi-simple scaler case) according to $\overline{\rho}$ (see the section immediately above 1.2; it may also be useful to compare this to Buzzard-Diamond-Jarvis [1] as a precursor to Gee-Kisin [2]). In fact, Kisin's proof works verbatim for the inequality (see Proposition 1.7.13):

$$e(R_{\mathrm{cr}}^{\square,\psi}(k,\tau,\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut,cr}}(k,\tau,\overline{\rho}) \stackrel{\mathrm{def}}{=} \sum_{n\in\{0,1,\ldots,p-1\}, m\in\{0,1,\ldots,p-2\}} a_{n,m,\mathrm{cr}}\mu_{n,m}(\overline{\rho}),$$

where $a_{n,m,\mathrm{cr}}$ is the multiplicity of $\sigma_{n,m}$ in the semi-simplification of $\overline{L}_{k,\tau,\mathrm{cr}}$. With that in mind, we shall only deal with the former.

2.3 Proof

Given a pseudo-deformation t of $\operatorname{tr}\overline{\rho}$, let I_t denote the kernel of the corresponding map $R^{\mathrm{ps}}(\operatorname{tr}\overline{\rho}) \to \mathcal{O}_t$. Let I denote the intersection of all I_t where t is the the trace of a representation of type $\mathcal{D}$; then there exists a countable set $U_{\mathcal{D}}$ [1] of pseudo-deformations of $\overline{r}$ such that $I = \bigcap_{t\in U_{\mathcal{D}}} I_t$.

Definition. We say that $\overline{\rho}$ is

(**Irr**) if it is absolutely IRReducible;
($\mathbf{NT}_{\chi_1,\chi_2}$) if it is a Non-Trivial extension of a character $\chi_2 : G \to \mathbf{F}^\times$ by a character $\chi_1 : G \to \mathbf{F}^\times$ such that χ_1 and χ_2 are distinct and $\chi_1/\chi_2 \notin \{\omega_{\mathrm{cyclo}}^{\pm 1}\}$;
($\mathbf{T}_{\chi_1,\chi_2}$) if it is a direct sum (i.e. a 'Trivial' extension) of distinct characters χ_1 and χ_1 such that $\chi_1/\chi_2 \notin \{\omega^{\pm 1}\}$;
($\mathbf{S}^3$) if it has Scalar Semi-Simplification.

We will show firstly that

$$e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})/\pi) \le \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho})$$

case-by-case, and then compare $e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi)$ and $e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})/\pi)$ to deduce the inequality. In the reducible case, one more or less has to prove inequalities over different components of $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$, and its geometry, in particular, understanding its irreducible components is crucial in Kisin's approach.

If $\overline{\rho} : G_{\mathbf{Q}_p} \to GL_2(E)$ is *indecomposable*, i.e., not reducible split, define

$$\mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho}) \stackrel{\mathrm{def}}{=} \sum_{n\in\{0,1,\dots,p-1\},m\in\{0,1,\dots,p-2\}} a_{n,m}\mu_{n,m,\mathrm{BDJ}}(\overline{\rho})$$

by setting $\mu_{n,m,\mathrm{BDJ}}(\overline{\rho}) = 0$ if $\mu_{n,m}(\overline{\rho}) = 0$ and $\mu_{n,m,\mathrm{BDJ}}(\overline{\rho}) = 1$ otherwise; in the light of the Buzzard-Dimaond-Jarvis conjecture, $\mu_{n,m,\mathrm{BDJ}}(\overline{\rho}) = 1$ precisely when BDJ [1] predicts Serre weights for such $\overline{\rho}$.

Lemma 2 *In view of computing e_Σ, let $\Sigma = \{\overline{\rho}\}$ if $\overline{\rho}$ is (**Irr**), and let $\Sigma = \{\omega_{\mathrm{cyclo}}^{n+1+m}\operatorname{unr}(\lambda_1\lambda_2)\}$ if $\overline{\rho} \sim \begin{pmatrix}\omega_{\mathrm{cyclo}}^{n+1}\operatorname{unr}(\lambda_1) & * \\ 0 & \operatorname{unr}(\lambda_1^{-1})\end{pmatrix} \otimes \omega_{\mathrm{cyclo}}^{m}\operatorname{unr}(\lambda_2)$, where $0 \le n, m \le p-2$ are integers and where $\operatorname{unr}(\lambda)$ is the unramified character of $G_{\mathbf{Q}_p}$ sending the geometric Frobenius to $\lambda \in \mathbf{F}^\times$. Assume, furthermore, that if $\overline{\rho}$ is reducible, $n = 0$, and $\lambda = \pm 1$, then $*$ is a peu ramifié extension. Then*

$$e_\Sigma(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})/\pi) \le \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})$$

unless $\overline{\rho}$ is ($\mathbf{S}^3$) in which case

$$e_\Sigma(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{\mathcal{D}}(\operatorname{tr}\overline{\rho})/\pi) \le 2\mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho}).$$

Sketch of proof. This is Lemma 1.6.8. Let $G = GL_2(\mathbf{Q}_p), K = GL_2(\mathbf{Z}_p)$ and Z the centre of G. Let P denote the set of representations $\pi(r,\lambda,\chi)$ such that $\mathbf{V}(\pi(r,\lambda,\chi))$ is a Jordan-Holder factor in $\overline{\rho}$. Since $\overline{L}_{k,\tau}$ is finite-dimensional over $\mathbf{F}$, there exists a filtration of KZ-subspaces $\{0\} = L_0 \subset L_1 \subset \cdots \subset L_n = \overline{L}_{k,\tau}$ such that every quotient L_{i+1}/L_i is an irreducible KZ-module. As it is irreducible, we may and will suppose that $L_{i+1}/L_i = \operatorname{Sym}^r\mathbf{F}^2 \otimes \chi \circ \det$ for some character $\chi : \mathbf{Q}_p^\times \to \mathbf{F}^\times$ such that $\chi|_{\mathbf{Z}_p^\times} = \det^s$. By some general commutative algebra results about e_Σ, it suffices to prove that, for Σ as defined above,

$$e_\Sigma(\text{c-Ind}_{KZ}^G L_{i+1}/L_i, R_i) = \mu_{r,s,\mathrm{Aut,BDJ}}(\overline{\rho})$$

unless $\overline{\rho}$ is ($\mathbf{S}^3$) in which case

$$e_\Sigma(\text{c-Ind}_{KZ}^G L_{i+1}/L_i, R_i) = 2\mu_{r,s,\mathrm{Aut,BDJ}}(\overline{\rho}),$$

where R_i denote the image of $R^{\mathrm{ps}}(\overline{\rho})$ in $\operatorname{End}(\mathbf{V}((\text{c-Ind}_{KZ}^G L_{i+1}/L_i)_P))$.

[1] It is denoted by U_0 in [4]

Firstly, observe that, with r and χ fixed as above, $\mu_{r,s,\mathrm{Aut,BDJ}}(\overline{\rho}) \neq 0$ if and only if there exists $\lambda \in \mathbf{F}$ such that $\mathbf{V}(\pi(r,\lambda,\chi))$ *is* the element in Σ; this is not true in the peu-ramifié case without the condition in the assertion.

If $\mu_{r,s,\mathrm{Aut,BDJ}}(\overline{\rho}) = 0$, it follows from Lemma 1.6.5 that $\mathbf{V}((\text{c-Ind}_{KZ}^{G} L_{i+1}/L_i)_P)$ has no sub-quotients isomorphic to Σ; hence $e_\Sigma = 0$.

If $\mu_{r,s,\mathrm{Aut,BDJ}}(\overline{\rho}) \neq 0$, it follows from Lemma 1.6.5 that

$$e_\Sigma(\mathbf{V}((\text{c-Ind}_{KZ}^{G} L_{i+1}/L_i)_P), R_i) = e_\Sigma(\mathbf{V}(\Pi(r,\lambda,\chi)), R_i).$$

The R^{ps}-module structure on $\mathbf{V}(\Pi(r,\lambda,\chi))$ is explicit and computable; and the RHS is 1 if $\overline{\rho}$ is not $(\mathbf{S}^3)$ while it is 2 if it is $(\mathbf{S}^3)$. □

Definition. An irreducible component Z of $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})[1/p]$ is said to be of *irreducible type* if the generic point of Z corresponds to an absolutely irreducible representation. If it is not, Z is said to be of *reducible type*.

In the reducible case, one can specify more. Suppose that the semi-simplification of $\overline{\rho}$ is the direct sum of characters χ_1 and χ_2. Let t be a closed point of an irreducible component Z of reducible type. The corresponding representation V_t is reducible (since the HT weights are distinct), and its semi-simplification is the direct sum of characters $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$. Which, we may and will assume, reduce respectively to χ_1 and χ_2. We say that t is *of type* χ_1 (resp. χ_2) if $\boldsymbol{\chi}_1$ (resp. $\boldsymbol{\chi}_2$) is the character of a one-dimensional *sub*space (as opposed to its one-dimensional quotient) of V_t. The following lemma shows that all points of Z of reducible type is either of type χ_1 and χ_2 and we say the component Z is *of type* χ_1 or χ_2 accordingly.

Lemma 3 *One knows exactly* when *and* how *there can be an irreducible component of* $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})[1/p]$ *of reducible type.*

Proof. This is Lemma 1.6.13 of [4]. □

Proposition 4 *If* $\overline{\rho}$ *is* ($\mathbf{Irr}$)*, then*

$$e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}).$$

If $\overline{\rho}$ *is* ($\mathbf{NT}_{\chi_1,\chi_2}$)*, choose* $U_{\mathcal{D},\mathrm{irr},\chi_1}$ *so that* $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr},\chi_1}}(\operatorname{tr}\overline{\rho}) \subset \operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$ *is the Zariski closure of the union of the components of irreducible type* and *reducible* χ_1 *type. Then*

$$e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr},\chi_1}}(\operatorname{tr}\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}).$$

If $\overline{\rho}$ *is* ($\mathbf{S}^3$)*, let* $U_{\mathcal{D},\mathrm{irr}} \subset U_{\mathcal{D}}$ *denote a dense subset of points on the components of* irreducible type *in* $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$ *and* C_{red} *denote the set of components of reducible type. Then*

$$e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}(\operatorname{tr}\overline{\rho})/\pi) + |C_{\mathrm{red}}| \leq \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho}).$$

Sketch of Proof.

($\mathbf{Irr}$): This is Proposition 1.6.10. By definition, $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$ is a quotient of $R^{\mathrm{ps}}(\operatorname{tr}\overline{\rho}) \simeq R(\overline{\rho})$ (see 1.4.4 (1)); hence 'by specialisation of the universal deformation $R(\overline{\rho})$-module', there exists a rank 2 free $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$-module $M(U_{\mathcal{D}})$ on which $G_{\mathbf{Q}_p}$ acts.

For g in $G_{\mathbf{Q}_p}$, let

$$P_g(X) = X^2 - T(g)X + (T(g)^2 - T(g^2))/2$$

where $T : G_{\mathbf{Q}_p} \to R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$ is the universal pseudo-deformation of $\operatorname{tr}\overline{\rho}$; it is the characteristic polynomial of $M(U_{\mathcal{D}})$, and $P_g(g) = 0$ on $V(U_{\mathcal{D}})$ according to the action of $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$ on $V(U_{\mathcal{D}})$ (see Lemma 1.6.3).

It then follows from an algebra lemma (Lemma 1.6.11) that there is an injection:

$$M(U_{\mathcal{D}}) \otimes_{R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho}),\eta} \operatorname{Frac} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho}) \longrightarrow V(U_{\mathcal{D}}) \otimes_{R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho}),\eta} \operatorname{Frac} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$$

at any generic point η of $\operatorname{Spec} R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$. By 'clearing the denominators', one can then establish that there is an injection homomorphism of $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\operatorname{tr}\overline{\rho})$-modules:

$$(1)\ M(U_{\mathcal{D}}) \longrightarrow V(U_{\mathcal{D}}).$$

If we let $\Sigma = \{\overline{\rho}\}$, unraveling the definition,

$$(2)\ e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi) = e_\Sigma(M(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})).$$

Combining,

$$\begin{array}{rcl} e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi) & = & e_\Sigma(M(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho}))\ \text{(by (2))} \\ & \leq & e_\Sigma(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho}))\ \text{(by (1) and Corollary 1.3.4)} \\ & \leq & \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})\ \text{(by 1.6.8, or Lemma 2 above)} \\ & = & \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}). \end{array}$$

($\mathbf{NT}_{\chi_1,\chi_2}$): This is Proposition 1.6.15. For brevity, let U denote $U_{\mathcal{D},\mathrm{irr},\chi_1}$. Let I_{irr} (resp. I_{χ_1}) denote the ideal of $R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})$ corresponding to the component of irreducible type (resp. of reducible χ_1 type). Let $V(U)_{\mathrm{irr}} = V(U)/I_{\mathrm{irr}}V(U)$ and $V(U)_{\chi_1} = V(U)/I_{\chi_1}V(U)$.

As χ_1 and χ_2 are distinct and $\chi_1/\chi_2 \notin \{\omega^{\pm 1}\}$, $\mathrm{Ext}^1(\chi_2,\chi_1)$ is one-dimensional and there exists a free rank 2 $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$-module $M(U)$ (see Corollary 1.4.7) on which $G_{\mathbf{Q}_p}$ acts; furthermore, $M(U)/I_{\chi_1}M(U)$ has a $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$-line L_{χ_1} on which $G_{\mathbf{Q}_p}$ acts by a character lifting χ_1.

Firstly observe that

$$(1)\ e_{\{\chi_1\}}(R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/(I_{\mathrm{irr}},\pi), R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) \leq e_{\{\chi_1\}}(V(U)_{\mathrm{irr}}/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi);$$

this follows as in the case (**Irr**) above. Similarly,

$$(2)\ e_{\{\chi_1\}}(R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/(I_{\chi_1},\pi), R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) = e_{\{\chi_1\}}(L_{\chi_1}/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) \leq e_{\{\chi_1\}}(V(U)_{\chi_1}/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi)$$

Finally observe that (3) $V(U) \longrightarrow V(U)_{\mathrm{irr}} \oplus V(U)_{\chi_1}$ is an isomorphism at the generic points of $\mathrm{Spec}\, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})$. Then

$$\begin{array}{rcl} e(R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) & = & e(R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/(I_{\mathrm{irr}},\pi), R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) + e(R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/(I_{\chi_1},\pi), R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) \\ & \leq & e_{\{\chi_1\}}(V(U)_{\mathrm{irr}}/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi) + e_{\{\chi_1\}}(V(U)_{\chi_1}/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi)\ \text{(by (1) and (2))} \\ & = & e_{\{\chi_1\}}(V(U)/\pi, R^{\mathrm{ps}}_U(\mathrm{tr}\,\overline{\rho})/\pi)\ \text{(by (3) and 1.3.4 (2))} \\ & \leq & \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})\ \text{(by 1.6.8, i.e., Lemma 2 above)} \\ & = & \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}). \end{array}$$

($\mathbf{S}^3$): This is Proposition 1.6.18.

Firstly observe that, for Σ as defined in 1.6.8, or Lemma 2 above,

$$(1)\ e(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi) = e_\Sigma(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)) \leq 2\mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho}).$$

Secondly,

$$(2)\quad \begin{array}{rcl} e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}(\mathrm{tr}\,\overline{\rho})/\pi) & \leq & e(V(U_{\mathcal{D},\mathrm{irr}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2 \\ & = & e(V(U_{\mathcal{D},\mathrm{irr}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2 \end{array}$$

For every $Z \in C_{\mathrm{red}}$, let $U_{\mathcal{D},Z} \subset U_{\mathcal{D}}$ denote a Zariski dense set of points of Z. Then

$$(3)\ 1 \leq e(V(U_{\mathcal{D},Z})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2.$$

Combining,

$$\begin{array}{rcl} e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}(\mathrm{tr}\,\overline{\rho})/\pi) + |C_{\mathrm{red}}| & \leq & e(V(U_{\mathcal{D},\mathrm{irr}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2 + \sum_{Z\in C_{\mathrm{red}}} 1\ \text{(by (2))} \\ & \leq & e(V(U_{\mathcal{D},\mathrm{irr}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2 + \sum_{Z\in C_{\mathrm{red}}} e(V(U_{\mathcal{D},Z})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2\ \text{(by (3))} \\ & = & e(V(U_{\mathcal{D}})/\pi, R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)/2 \\ & \leq & \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})\text{(by (1))}. \end{array}$$

□

The following lemma is the first step towards comparing $e(R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/\pi)$ and $e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi)$.

Lemma 5 *The universal property of $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})$ defines a map*

$$R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho}) \longrightarrow R^{\square,\psi}(k,\tau,\overline{\rho})$$

and it factors through $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})$; if $\mathrm{End}(\overline{\rho}) \subseteq \mathbf{F}$, then

$$R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho}) \longrightarrow R^{\psi}(k,\tau,\overline{\rho})$$

factors through $R^{\psi}_{U_{\mathcal{D}}}(k,\tau,\overline{\rho})$. If, in particular, $\overline{\rho}$ is ($\mathbf{NT}_{\chi_1,\chi_2}$), then it factors through $R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr},\chi_1}}(\mathrm{tr}\,\overline{\rho})$ (as defined in Proposition 4 above).

Proof. This is Lemma 1.7.1. It is clear by definition. □

Corollary 6 *If $\overline{\rho}$ is either ($\mathbf{Irr}$) or ($\mathbf{NT}_{\chi_1,\chi_2}$), then*

$$e(R^{\psi}(k,\tau,\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}).$$

Proof. Let $U = U_{\mathcal{D}}$ if $\overline{\rho}$ is ($\mathbf{Irr}$) and let $U = U_{\mathcal{D},\mathrm{irr},\chi_1}$ if $\overline{\rho}$ is ($\mathbf{NT}_{\chi_1,\chi_2}$). Then

$$e(R^{\psi}(k,\tau,\overline{\rho})/\pi) \leq e(R^{\mathrm{ps}}_{U}(\mathrm{tr}\,\overline{\rho})/\pi) \leq \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}),$$

where the first inequality follows from the preceding lemma while the second inequality follows from 1.6.10 if $\overline{\rho}$ is ($\mathbf{Irr}$) and 1.6.15 if $\overline{\rho}$ is ($\mathbf{NT}_{\chi_1,\chi_2}$). □

This corollary leaves us the two more cases ($\mathbf{T}_{\chi_1,\chi_2}$) and ($\mathbf{S}^3$). In order to understand these cases, one has to understand more about $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})$. Suppose that $\overline{\rho}$ is either ($\mathbf{T}_{\chi_1,\chi_2}$) or ($\mathbf{S}^3$); in particular, $\overline{\rho}$ is reducible.

From (1.5.11), there is a map

$$\theta : R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho}) \to \mathbf{F}[[S]]$$

corresponding to the deformation $\mathbf{V}(\Pi(r,\lambda,\chi))$ over $\mathbf{F}[[S]]$ (where $S = T - \lambda$) of a Jordan-Holder factor $\mathbf{V}(\pi(r,\lambda,\chi))$ of $\overline{\rho}$; furthermore, since $\overline{\rho}$ is reducible, θ depends only on the semi-simplification of $\overline{\rho}$ and not on (r,λ,χ).

Definition. Let $J = \ker\theta$.

Definition. Let $R^{\mathrm{ord}} = R^{\square,\psi}(\overline{\rho})/J$ where by 'J' we mean the image of J by $R^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho}) \to R^{\square,\psi}(\overline{\rho})$.

Lemma 7 *If $\overline{\rho}$ is ($\mathbf{T}_{\chi_1,\chi_2}$), then $\mathrm{Spec}\,R^{\mathrm{ord}}$ has two components each of which is formally smooth over $\mathbf{F}$ and dominates $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/J$.*

If, on the other hand, $\overline{\rho}$ is ($\mathbf{S}^3$) and $\overline{\rho} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \otimes \chi$ where $\chi : G_{\mathbf{Q}_p} \to \mathbf{F}^{\times}$ satisfies $\chi^2 = \psi\chi_{\mathrm{cycl}}$, then $\mathrm{Spec}\,R^{\mathrm{ord}}$ is irreducible, generically reduced and dominates $R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})/J$. If, furthermore, $* = 0$, then $(R^{\mathrm{ord}})^{\mathrm{red}}$ is formally smooth over $\mathbf{F}$.

Proof. This is Lemma 1.7.4 if $\overline{\rho}$ is ($\mathbf{T}_{\chi_1,\chi_2}$), while it is Lemma 1.7.5 if $\overline{\rho}$ is ($\mathbf{S}^3$). We leave it to the reader to check their proofs. □

Definition. Let $U \subset U_{\mathcal{D}}$ be a set of closed points of $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})$ whose closure is a non-empty collection of irreducible components. Define

$$R_U$$

to be the image of $R^{\square,\psi}(k,\tau,\overline{\rho})$ in $R^{\square,\psi}(k,\tau,\overline{\rho}) \otimes_{R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})} R^{\mathrm{ps}}_{U}(\mathrm{tr}\,\overline{\rho})[1/p]$).

A point t of $\mathrm{Spec}\,R_U$ defined over a finite extension E_t of E corresponds to a two-dimensional representation of $G_{\mathbf{Q}_p}$ over E_t of type $\mathcal{D}$ whose trace reduces to $\mathrm{tr}\,\overline{\rho}$, and it lies on an irreducible component in U.

Lemma 8 *If $\overline{\rho}$ is $(\mathbf{T}_{\chi_1,\chi_2})$ or $(\mathbf{S}^3)$, then*

$$e(R_U/\pi) \le e(R_U^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})/\pi)e(R^{\mathrm{ord}}).$$

If $\overline{\rho}$ is $(\mathbf{T}_{\chi_1,\chi_2})$ and U consists of type χ_1, then

$$e(R_U/\pi) \le e(R_U^{\mathrm{ps}}(\mathrm{tr}\,\overline{\rho})/\pi).$$

Proof. This is Lemma 1.7.7. □

Proposition 9 *If $\overline{\rho}$ is $(\mathbf{T}_{\chi_1,\chi_2})$, then*

$$e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi) \le \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}).$$

If $\overline{\rho}$ is $(\mathbf{S}^3)$ and $\overline{\rho} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \otimes \chi$, then

$$e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi) \le e(R^{\mathrm{ord}})\mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho}) = \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}).$$

Sketch of proof.

$(\mathbf{T}_{\chi_1,\chi_2})$: This is Proposition 1.7.8. Let $U_{\mathcal{D},\mathrm{irr}}$ be a subset of $U_{\mathcal{D}}$ such that $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}(\mathrm{tr}\,\overline{\rho})$ is the union of components of $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})$ of irreducible type; and let $U_{\mathcal{D},\mathrm{red},\chi_1}$(resp. $U_{\mathcal{D},\mathrm{red},\chi_2}$) be a subset of $U_{\mathcal{D}}$ such that $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_1}}(\mathrm{tr}\,\overline{\rho})$ (resp. $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_2}}(\mathrm{tr}\,\overline{\rho})$) is the union of components of $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})$ of reducible χ_1 (resp. χ_2) type. Note that $U_{\mathcal{D},\mathrm{red},\chi_1}(\mathrm{tr}\,\overline{\rho})$ and $U_{\mathcal{D},\mathrm{red},\chi_2}(\mathrm{tr}\,\overline{\rho})$ are different from those 'with *irr* in place of *red*' as appeared in Proposition 4 above, i.e., $U_{\mathcal{D},\mathrm{irr},\chi_1}(\mathrm{tr}\,\overline{\rho})$ and $U_{\mathcal{D},\mathrm{irr},\chi_2}(\mathrm{tr}\,\overline{\rho})$ respectively.

By 1.7.4 and 1.7.7,

$$(1)\quad \begin{aligned} e(R_{U_{\mathcal{D},\mathrm{irr}}}/\pi) &\le e(R^{\mathrm{ord}})e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}/\pi) \text{ (by 1.7.7)} \\ &= 2e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}/\pi) \text{ (by 1.7.4).} \end{aligned}$$

Let $\overline{\rho}_{\chi_1,\chi_2}$ (resp. $\overline{\rho}_{\chi_2,\chi_1}$) denote a non-trivial extension of χ_2 by χ_1 (resp. χ_1 by χ_2). Then it follows from 1.7.7. that

$$(2)\quad \begin{aligned} e(R_{U_{\mathcal{D},\mathrm{red},\chi_1}}/\pi) &\le e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_1}}/\pi); \\ e(R_{U_{\mathcal{D},\mathrm{red},\chi_2}}/\pi) &\le e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_2}}/\pi). \end{aligned}$$

Combining,

$$\begin{aligned} e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi) &= e(R_{U_{\mathcal{D},\mathrm{irr}}}/\pi) + e(R_{U_{\mathcal{D},\mathrm{red},\chi_1}}/\pi) + e(R_{U_{\mathcal{D},\mathrm{red},\chi_2}}/\pi) \\ &\le 2e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}/\pi) + e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_1}}/\pi) + e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_2}}/\pi) \text{ (by (1) and (2))} \\ &= \Big(e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}/\pi) + e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_1}}/\pi)\Big) + \Big(e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}/\pi) + e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{red},\chi_2}}/\pi)\Big) \\ &= e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr},\chi_1}}/\pi) + e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr},\chi_2}}/\pi) \\ &\le \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}_{\chi_1,\chi_2}) + \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}_{\chi_2,\chi_1}) \\ &= \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}). \end{aligned}$$

$(\mathbf{S}^3)$: This is Proposition 1.7.10. Firstly observe that

$$(1)\ \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}) = \mu_{p-2,s}(\overline{\rho})\mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})$$

where $\chi|_{I_{\mathbf{Q}_p}} = \omega^s_{\mathrm{cyclo}}$, and $\mu_{p-2,s}(\overline{\rho}) = e(R^{\mathrm{ord}})$.

We may and will suppose henceforth that the twist χ is trivial. For a component Z of $\mathrm{Spec}\,R^{\mathrm{ps}}_{U_{\mathcal{D}}}(\mathrm{tr}\,\overline{\rho})[1/p]$ of reducible type, let $U_{\mathcal{D},Z}$ denote a Zariski dense set of points in Z. Then

$$(2)\ e(R_{U_{\mathcal{D},Z}}/\pi) = e(R^{\mathrm{ord}}).$$

This is not exactly straightforward (see Kisin's proof of Proposition 1.7.10). Combining,

$$\begin{aligned} e(R^{\square,\psi}(k,\tau,\overline{\rho})/\pi) &= e(R_{U_{\mathcal{D},\mathrm{irr}}}) + \textstyle\sum_{Z\in C_{\mathrm{red}}} e(R_{U_{\mathcal{D},Z}}/\pi) \\ &\le \Big(e(R^{\mathrm{ps}}_{U_{\mathcal{D},\mathrm{irr}}}) + \textstyle\sum_{Z\in C_{\mathrm{red}}} 1\Big) e(R^{\mathrm{ord}}) \text{ (by 1.7.7, } U = U_{\mathcal{D},\mathrm{irr}}, U_{\mathcal{D},Z} \text{ and (2))} \\ &\le \mu_{\mathrm{Aut,BDJ}}(k,\tau,\overline{\rho})e(R^{\mathrm{ord}}) \text{ (by 1.6.18)} \\ &= \mu_{\mathrm{Aut}}(k,\tau,\overline{\rho}) \text{ (by (1)),} \end{aligned}$$

where $U_{\mathcal{D},\mathrm{irr}}$ denotes a Zariski dense set of points on the components of irreducible type and C_{red} denotes the set of components of reducible type. □

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