Hida-Coleman theory (Draft 29/09/15)

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1 Introduction

I've been asked to give a series of lectures on Hida-Coleman theory of p-adic modular forms. Here are my notes.

To that end, I should like to list my references: Hida's 'Elementary theory of L-functions and Eisenstein series' (CUP) and 'On p-adic Hecke algebras over totally real fields' (Annals); Coleman's 'p-adic Banach spaces and families of modular forms' (Inventiones) and 'Classical and overconvergent modular forms' (Inventiones).

I was encouraged not to talk about Galois representations, but they are, really, inseparable (to say the least) for the development of theory of p-adic modular forms. If you are interested, the following list might be useful and accessible: Hida's 'Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms' (Inventiones) and 'Nearly ordinary Hecke algebras and Galois representations of several variables' (Proceedings to a JAMI conference); Mazur-Wiles 'On p-adic analytic families of Galois representations' (Compositio). A proper reference for p-adic geometry would be BGR 'Non-archimedean analysis', but Schneider's notes 'Basic notions of rigid analytic geometry' might just teach you enough to understand, at least, claims I shall be making.

I learned almost everything in here by listening to my advisor Kevin Buzzard. Anything he's ever written about p-adic modular forms is careful and informative, so I suggest you study them if you come across.

2 A completely unnecessary but maybe useful remarks for pedants?

For an element γ of $\mathrm{GL}_2^+(\mathbf{R})$, define '$f|_k\gamma$' to be $(\det\gamma)(cz+d)^{-k}f(\gamma z)$. The power '1' of det matters significantly in considering modular forms of levels defined by open compact subs of GL_2, rather than of classical SL_2. Often Shimura, and Hida's 'Iwasawa modules...' and 'Galois representations...' papers, choose it to be $k/2$ for making automorphic representations 'unitarizable'. On the other hand, Hida's other papers, and Buzzard-Taylor for example, essentially choose it to be 1. This difference is apparent in how one normalises a 'Λ-algebra structure' and, as a result, in 'specialisations' of weight characters. More precisely, it is defined by $|_k\gamma$ 'augmented' by the k-th power of γ_p for $\gamma=\gamma_p\gamma_N\in\mathbf{Z}_p^\times\times(\mathbf{Z}/N\mathbf{Z})^\times$ in the former (See Hida's 'Galois representations...' p.549), while via the diamond operator $\langle\gamma\rangle=\chi^{2-k}(\gamma)(|_k\gamma)$ in the latter. I've never got my head around Wiles' 'On ordinary...' paper, where he follows Shimura to choose $k/2$ but specialisation is $k-2$-power... perhaps it is how $\Lambda\to T^{\mathrm{o}}(N;O)$ is normalised?

3 Hida theory

Let $k\geq 1$ and $M\geq 1$ be integers. Let $S_k(\Gamma_1(M);\mathbb{C})$ denote the finite-dimensional $\mathbb{C}$-vector space of cusp forms of weight k and level $\Gamma_1(M)$.

For a modular form f, let $a(\nu,f)$ denote the Fourier coefficients.

For every $\mathbf{Z}$-algebra O in $\mathbb{C}$, let $S_k(\Gamma_1(M);O)$ denote the sub of f in $S_k(\Gamma_1(M);\mathbb{C})$ such that $a(\nu,f)\in O$.

It is well-known:

$$S_k(\Gamma_1(M);O)\otimes_O \mathbb{C}\simeq S_k(\Gamma_1(M);\mathbb{C}).$$

If O is a $\mathbf{Z}$-algebra but not necessarily in $\mathbb{C}$, define $S_k(\Gamma_1(M);O)$ to be $S_k(\Gamma_1(M);\mathbf{Z})\otimes_{\mathbf{Z}} O$.

Oecall that $S_k(\Gamma_1(M);\mathbb{C})$ comes equipped with action of $(\mathbf{Z}/M\mathbf{Z})^\times$ by the diamond operator $\langle\ \rangle$, T_V (resp. U_V) for a prime V not dividing (resp. dividing) M. Also $S_V=V^{k-2}\langle V\rangle$ for every prime V not dividing M.

For every integer $N\geq 1$, we define $T(N)$, or T_N by: $T(RS)=T(R)T(S)$ for coprime integers R,S, $\sum_{\rho=0}^{\infty}T(V^\rho)X^\rho=(1-T(V)X+VS_VX^2)^{-1}$ for V not dividing M; and $T(V^\rho)=U_V^\rho$ for V dividing M.

A classical result:

Lemma 1 *For f in $S_k(\Gamma_1(M);\mathbb{C})$,*

$$a(1,f|T_N)=a(N,f)$$

Proof. Hida Corollary 4.2. □

Let $T_k(M;\mathbf{Z})$ denote the subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(M);\mathbb{C}))$ generated over $\mathbf{Q}$ by the T_N's. For any $\mathbf{Z}$-algebra O, let $T_k(M;O)=T_k(M;\mathbf{Z})\otimes_{\mathbf{Z}} O$.

Lemma 2 *$T_k(M;O)$ is a flat module O of fine type (i.e. finitely generated over O). If O is finite flat over $\mathbf{Z}_p$, it is finite free.*

Proof. The first assertion is Hida's Theorem 3.1. The second assertion is simply a commutative algebra exercise (Matsumura Theorem 7.10). □

Theorem 3 *For every sub $\mathbf{Z}$-algebra O in $\mathbb{C}$, $S_k(\Gamma_1(M);O)$ is stable under T_N for every N, hence stable under $T_k(M;O)$. As a result, the same assertion holds for O not necessarily in $\mathbf{C}$, and $T_k(M;O)$ acts on $S_k(\Gamma_1(M);O)$.*

Proof. It might be surprising that algebraic geometry is necessary to define 'integral structure' of modular forms. For f in $S_k(\Gamma_1(M),O)$, Shimura/Hida's formula asserts

$$a(\ell,f|T_N)=\sum|\mathbf{Z}/V\mathbf{Z}|a((\ell/V)(N/V),f|S_V)$$

the sum ranges over V satisfying $V|\ell,V|N$, and V coprime to M. Hence it suffices to check that $|\mathbf{Z}/V\mathbf{Z}|S_V$ acts on $S_k(\Gamma_1(M);O)$. To check this, as $S_k(\Gamma_1(M);O)=H^0(X_O,\omega^{\otimes k-2}\otimes\Omega^1)$, one calculates q-expansion coefficients of $|\mathbf{Z}/V\mathbf{Z}|S_V$ (which needs to be interpreted in terms of moduli) at Tate curves, and ascertain they are defined over O. □

Define a pairing

$$\langle\ ,\ \rangle: T_k(M;O)\times S_k(\Gamma_1(M);O)\longrightarrow O$$

by $\langle T,f\rangle=a(1,f|T)$.

By definition, $\langle TS,f\rangle=\langle S,f|T\rangle$ and one can deduce $T_k(M,O)$ is a commutative algebra with identity $T(1)$; and $\langle T_N,f\rangle=a(N,f)$.

Corollary 4 *If O is flat over $\mathbf{Z}$, then*

$$\mathrm{Hom}_O(T_k(M;O),O)\simeq S_k(\Gamma_1(M);O)$$

$$\mathrm{Hom}_O(S_k(\Gamma_1(M);O),O)\simeq T_k(M;O).$$

Note that, by 'Hom_O' I mean O-linear homomorphisms, and no more.

Proof. Firstly, suppose that O is a field L. As $S_k(\Gamma_1(M);L)$ and $T_k(M;L)$ is finite-dimensional over L, it suffices to check that the pairing is non-degenerate. Suppose $\langle T,f\rangle=0$ for every T. Then the lemma shows that $a(N,f)=a(1,f|T_N)=\langle T_N,f\rangle=0$, and therefore $f=0$.

On the other hand, suppose that $\langle T, f\rangle = 0$ for every f. Then $a(N, f|T) = a(1, f|TT_N) = a(1, f|T_N T) = \langle T, f|T_N\rangle = 0$ hence $T = 0$ as an operator.

Suppose that O is no longer a field. Since $S_k(\Gamma_1(M); O) = \bigcap_V S_k(\Gamma_1(M); O_V)$ for localisations O_V, we may assume that O is a local valuation ring with field of fractions L.

To prove $\mathrm{Hom}_O(T_k(M; O), O) \simeq S_k(\Gamma_1(M); O)$ for example, observe that, since $T_k(M; O)$ is finite over O, $T_k(M; O) \otimes_O L$ is a subalgebra of $T_k(M; L)$, hence any λ in $\mathrm{Hom}_O(T_k(M; O), O)$ extends to a L-linear map $T_k(M; L) \to L$. By the duality over L, there exists f in $S_k(\Gamma_1(M); L)$ such that $\lambda(T) = \langle T, f\rangle$ for every T. It therefore follows that, for every N, $a(N, f) = a(1, f|T_N) = \langle T_N, f\rangle = \lambda(T_N) \in O$ since $T_N \in T_k(M; O)$. Hence f lies in $S_k(\Gamma_1(M); O)$ indeed. □

Suppose that f is a non-trivial eigenform, i.e., $f|T_N = \lambda_N f$ for some $\lambda \in \mathbb{C}$, for every $N \geq 1$. Then the map sending T_N to λ_N defines a $\mathbb{C}$-algebra map $T_k(M; \mathbb{C}) \to \mathbb{C}$.

Observe that $\lambda_N \langle T(1), f\rangle = \langle T(1), f|T_N\rangle = \langle T_N, f\rangle = a(N, f)$. So, as f is non-trivial, $\langle T(1), f\rangle$ is non-zero. And it makes sense to consider $f/\langle T(1), f\rangle$ instead, and $\lambda_N = a(N, f)$ now holds for this new f. When $\lambda_N = a(N, f)$ holds, we call f a normalised eigenform.

Theorem 5 $\text{Alg-Hom}_O(T_k(M; O), L)$ *is isomorphic to the subspace of* $S_k(\Gamma_1(M); \mathbb{C})$ *of normalised eigenforms.*

Algebra O-homomorphisms are additionally 'multiplicative'.

M will be Np^ν where N is prime to p from now on. Suppose that $p \geq 5$. Suppose that O is finite flat over O.

Definition. As ν ranges over integers ≥ 1, $T_k(Np^\nu; O)$ form an inverse system, and let $T_k(Np^\infty; O)$ denote the limit. On the other hand, $S_k(Np^\nu; O)$ form a direct system, and let $S_k(Np^\infty; O)$ denote the limit.

The diamond operator gives rise to

$$\langle\ \rangle : (\mathbf{Z}/Np^\nu\mathbf{Z})^\times \longrightarrow T_k(Np^\nu; O)$$

which, in turn, gives rise to

$$(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times \longrightarrow T_k(Np^\infty; O).$$

Note that $(\mathbf{Z}/Np^\nu\mathbf{Z})^\times$ is nothing other than the (classical) ideal class group mod $Np^\nu\infty$. Observe that $\mathbf{Z}_p^\times \simeq (\mathbf{Z}/p\mathbf{Z})^\times \times (1 + p\mathbf{Z}_p)^\times$. The latter $(1 + p\mathbf{Z}_p)^\times$ is topologically generated by $u = 1 + p$ for example, and the completed group algebra $\Lambda = \mathbf{Z}_p[(1 + p\mathbf{Z}_p)^\times]$ is isomorphic:

$$\Lambda \simeq \mathbf{Z}_p[[\xi]]$$

by sending u to $1 + \xi$.

Geometrically, $\mathcal{W} = \mathrm{Sp}\,(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times$, often called the weight space, is a disjoint union of $|(\mathbf{Z}/pN\mathbf{Z})^\times|$-copies of the 'ball' $\mathrm{Sp}\,\Lambda$. Perhaps easier to fathom the idea by looking at $\overline{\mathbf{Q}}_p$-points: the $\overline{\mathbf{Q}}_p$-points of $\mathcal{W}$ corresponds to the characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times \simeq (\mathbf{Z}/Np\mathbf{Z})^\times \times (1 + p\mathbf{Z}_p^\times)^\times \to \overline{\mathbf{Q}}_p$; and each ball/component is indexed by its restriction to $(\mathbf{Z}/Np\mathbf{Z})^\times$, i.e., every pair of characters in the same component define the same character of $(\mathbf{Z}/Np\mathbf{Z})^\times$.

One can define a norm $|\ |$ on $S_k(Np^\infty; O)$ by $|f| = \sup_N |a(N, f)|$, and let $S_k^\wedge(Np^\infty; O)$ denote the completion with respect to this norm. When p inverted, this defines a Banach space and its elements are often called (holomorphic) p-adic cusp forms. Good references for theory of p-adic modular forms are Katz 'Higher congruences between modular forms' and Gouvea's thesis.

Theorem 6 *For every* $\mathbf{Z}_p$*-algebra* O*, there is a natural isomorphism as* $O \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[(\mathbf{Z}/Np\mathbf{Z})^\times]$*-algebras*

$$T_k(Np^\infty, O) \simeq T_2(Np^\infty; O)$$

sending T_N *to* T_N*, for every* $k \geq 2$*. Similarly for* $S_k^\wedge(Np^\infty; O)$*.*

Proof. Hard. □

Given this 'weight-independence' result, we shall call it $T(N;O)$ from now on. This is the (p-adic) Hecke algebra of p-adic modular forms (of tame level N).

For record:

Proposition 7 *The pairing* $\langle\ ,\ \rangle : T_k(Np^\infty;O) \times S_k^\wedge(Np^\infty;O) \to O$*, defined exactly the same as before, defines*

$$\mathrm{Hom}_O(T_k(Np^\infty;O),O) \simeq S_k^\wedge(\Gamma_1(Np^\infty);O)$$
$$\mathrm{Hom}_O(S_k^\wedge(\Gamma_1(Np^\infty);O),O) \simeq T_k(Np^\infty;O).$$

Define $T^{\mathrm{o}}(N;O)$ to the maximal direct summand of $T(N;O)$ on which T_p is invertible. This is what people call the Hida ordinary Hecke algebra (of tame level N). To define this, we firstly define the 'Hida idempotent' e on $T_k(Np^\nu;O)$ as follows: since $T = T_k(Np^\nu;O)$ is a flat O-module of finite type over O (Hida Theorem 3.1), it is a p-adically complete semi-local ring, and is a direct sum

$$T = \bigoplus_{\mathfrak{m}} T_{\mathfrak{m}}$$

of localised T at maximal ideals. W then write eT for the direct sum of all $T_{\mathfrak{m}}$ with $\mathfrak{m}$ ranging over all maximal ideals *not* containing $T(p)$. We define $T^{\mathrm{o}}(N;O)$ to be the inverse limit of $eT_k(Np^\nu;O)$ for any $k \geq 2$.

Theorem 8 $T^{\mathrm{o}}(N;O)$ *is torsion-free, and of finite-type as a* Λ*-module.*

Theorem 9 $T^{\mathrm{o}}(N;O)$ *is free of finite rank over* Λ*.*

Proposition 10 Alg-$\mathrm{Hom}_O(T_k^{\mathrm{o}}(Np^\nu;O),\overline{\mathbf{Q}}_p)$ *equals the set of normalised eigenform* f *in* $S_k(Np^\nu;O)$ *such that its slope* $\mathrm{val}(a(p,f))$ *is* 0*, where* val *denote the normalised valuation on* $\overline{\mathbf{Q}}_p$ *such that* $\mathrm{val}(p) = 1$*.*

f as in the proposition above is called a (Hida) ordinary form.

Suppose that O contains a p^ν-power roos of unity ζ. Let $k \geq 2$ be an integer.

A character χ of $\mathbf{Z}_p[(1+p\mathbf{Z}_p)^\times] \to O$ is determined by where u is sent to, and define it to be $(1+p)^{k-2}\zeta$. Then:

Theorem 11 $T^{\mathrm{o}}(N;O)/\ker\chi = T^{\mathrm{o}}(N;O)/(u-(1+p)^{k-2}\zeta)$ *is isomorphic to the maximal quotient of* $T_k^{\mathrm{o}}(Np^{\nu+1};O)$ *acting on the subspace of* $S_k(\Gamma_1(Np^{\nu+1});O)$ *with Dirichlet characters of* $(\mathbf{Z}/Np^{\nu+1}\mathbf{Z})^\times$*, when restricted to* $(\mathbf{Z}/p^\nu\mathbf{Z})^\times \simeq (1+p\mathbf{Z}_p)^\times/(1+p^\nu\mathbf{Z}_p)^\times$*, equals* χ*.*

By the duality theorems mentioned, this theorem asserts that a ordinary p-adic modular eigenform of weight $k \geq 2$ *is* a cusp modular eigenform. We shall see in Coleman-theory a result, an overconvergent p-adic modular eigenform of slope $< k-1$ is classical, that is a generalisation of this result.

Let K denote the field of fractions of Λ, and $\overline{K}$ denote its algebraic closure.

A Λ-algebra homomorphism

$$\mathcal{F} : T^{\mathrm{o}}(N;O) \longrightarrow \overline{K}$$

is often called a Λ-adic form. Observe that, there is an extension $K \subset L \subset \overline{K}$ through which $\mathcal{F}$ factors.

Consider $\mathrm{Spec}\,L$. Its $\overline{\mathbf{Q}}_p$-points correspond to Alg-$\mathrm{Hom}(L,\overline{\mathbf{Q}}_p)$, and we call a point P of $(\mathrm{Spec}\,L)(\overline{\mathbf{Q}}_p)$ *arithmetic* if its restriction to Λ maps u to $(1+p)^{k-2}\zeta$ for an integer $k \geq 2$ and for some p-power root ζ. In which case, the specialisation $\mathcal{F}_P = P \circ \mathcal{F} : T^{\mathrm{o}}(N;O) \to \overline{\mathbf{Q}}_p$ factors through $T^{\mathrm{o}}(N;O)/(u-(1+p)^{k-2}\zeta)$, and therefore defines a cusp eigenform $\sum_{N=1}^{\infty} \mathcal{F}(T_N)(P)q^N \in S_k(\Gamma_1(Np^{\nu+1});\overline{\mathbf{Q}}_p)$. In some sense $\mathcal{F}$ is a one-parameter family in $L[[q]]$ of ordinary cusp forms.

If I have time, explain concrete examples of Hida families from Kev's notes 'Examples of Hida families'. Hopefully freely available.

4 Quick recap about p-adic geometry

Let O be a complete, non-archimedean, valuation ring, with its field of fractions L. Let π be a uniformiser.

Suppose that K is an affinoid L-algebra. Define *semi-norm*, called the supremum semi-norm, $|\ |$ on $X = \operatorname{Sp} K = \operatorname{Max} K$ by $|f| = \sup_{\xi \in X} |f(\xi)|$ (note that ξ corresponds to a maximal ideal of K and K/ξ is a finite extension of L, and $|f(\xi)|$ denote the norm of the image of f in K/ξ).

$$R = \{f \in K \,|\, |f| \leq 1\} \supset S = \{f \in K \,|\, |f| < 1\}$$

and $\overline{R} = R/S$. For example, if $K = L\langle X_1, \ldots, X_N\rangle$ the Tate algebra, $R = O\langle X_1, \ldots, X_N\rangle$, which is the π-adic completion of $O[X_1, \ldots, X_N]$ whose elements are of the form $\sum_{\lambda_1,\ldots,\lambda_N} c_{\lambda_1,\ldots,\lambda_N} X_1^{\lambda_1} \cdots X_N^{\lambda_N}$, where $c_{\lambda_1,\ldots,\lambda_N} \in O$ such that $|c_{\lambda_1,\ldots,\lambda_N}| \to 0$ as $\lambda_1 + \cdots + \lambda_N \to \infty$, and $\overline{R} = k[X_1, \ldots, X_N]$.

One can show that
- R is a model of K, i.e., $R \otimes_O L \simeq K$;
- R is π-adically complete;
- R is topologically of finite type over O, i.e., $\simeq O\langle X_1, \ldots, X_N\rangle/I$ for some ideal I;
- R is O-flat;
- $\overline{R}$ is finite type k-algebra.

So it makes sense to define:

Definition. An O-algebra R is *admissible* if it is π-adically complete, flat over O, and topologically of finite type.

Definition. A formal scheme $\mathcal{X}$ over O s admissible if it is locally $\operatorname{Spf} R$ for an admissible O-algebra R.

Definition. $\operatorname{sp} : \mathcal{X} = \operatorname{Spf} R \to \overline{\mathcal{X}} = \operatorname{Spec} \overline{R}$ given by sending I to $(I \cap R)/(I \cap M)$. It surjects onto the closed points of $\overline{\mathcal{X}}$. This 'globalises'.

Definition. For an admissible formal scheme $\mathcal{X}$ over O, its admissible formal blow-up along I is:

$$\mathrm{BL}_{\mathcal{X}} = \lim_{N} \Pr \bigoplus_{\lambda=0}^{\infty} (I^{\lambda} \otimes_{O_{\mathcal{X}}} O_{\mathcal{X}}/\pi^N) \longrightarrow \mathcal{X}$$

Given affine admissible formal scheme $\mathcal{X} = \operatorname{Spf} R$ over O, $\mathcal{X}^{\mathrm{a}} = \operatorname{Sp} R \otimes_O L$ defines a rigid analytic space, because $R \otimes_O L$ is an affinoid L-algebra. For example, if $R \simeq O\langle X_1, \ldots, X_N\rangle/I$, then $R \otimes_O L \simeq L\langle X_1, \ldots, X_N\rangle/I$. This 'functor' $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{a}}$ 'globalises' (call the latter the Raynaud generic fibre of $\mathcal{X}$), and gives the equivalence of categories between

- the category of quasi-compact, admissible formal schemes over O, localised by admissible formal blow-ups;
- the category of quasi-compact (i.e admits a finite admissible affinoid covering), quasi-separated (i.e., the diagonal morphism $\Delta : \mathcal{X}^{\mathrm{a}} \to \mathcal{X}^{\mathrm{a}} \times_L \mathcal{X}^{\mathrm{a}}$ is a quasi-compact morphism; See Bosch's lecture notes 1.16, Proposition 4 that the Berthelot's definition that the intersection of two affonoids is an affinoid is weaker), rigid analytic spaces over L.

In particular, $\mathrm{BL}_{\mathcal{X}} \to \mathcal{X}$ gives rise to an isomorphism of rigid spaces $(\mathrm{BL}_{\mathcal{X}})^{\mathrm{a}} \simeq \mathcal{X}^{\mathrm{a}}$.

5 Coleman theorey

Let Y denote $\Gamma_1(N)\backslash\mathbf{H} = \mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbf{R}_+^{\times}\mathrm{SO}_2(\mathbf{R})$, and let X denote the Borel-Serre compactification over $\mathbf{C}$.

Let $\omega^{\otimes k}$ denote the cotangent bundle

$$\Gamma_1(N)\backslash(\mathbf{H}\times\mathbb{C}) \longrightarrow \Gamma_1(N)\backslash\mathbf{H} = Y$$

where γ in $\Gamma_1(N)$ acts via $\gamma(z,\xi) = ((az+b)/(cz+d), (cz+d)^k\xi)$. Borel-Serre theory establishes that $\omega^{\otimes k}$ extends to X. Parenthetically, the standard representation $\mathbb{C}^2$ corresponds to the relative de Rham cohomology sheaf over Y, which extends over X.

It is a standard exercise to check

$$H^0(X,\omega^{\otimes k}) \simeq M_k(\Gamma_1(N);\mathbb{C})$$

and

$$H^0(X,\omega^{\otimes k-2}\otimes\Omega^1) \simeq S_k(\Gamma_1(N);\mathbb{C}).$$

One can see Y as a moduli space of the set of isomorphism classes of elliptic curves $\mathbb{C}/\langle\mathbf{Z}+\mathbf{Z}\tau\rangle$ with level structure $1/N$.

The Tate curve $\mathbb{G}/q^{\mathbf{Z}}$ over $\mathbf{Z}((q))$ defines

$$\operatorname{Spec}\mathbf{Z}((q)) \to Y$$

corresponding to $(\mathbb{G}/q^{\mathbf{Z}}, \mu_N \hookrightarrow \mathbb{G})$ extends to

$$\operatorname{Spec}\mathbf{Z}[[q]] \to X$$

and the pul-back of $f \in H^0(X,\omega^{\otimes k})$ to $\mathbf{Z}[[q]]$ define the q-expansion of f.

One may define Hecke operators T_Q/U_Q via degeneracy maps

$$\pi_1,\pi_2 : X_{\mathrm{I}} \longrightarrow X$$

for the Iwahori subgroup I at Q, and a map of morphisms $\pi_2^*\mathcal{F} \to \pi_1^*\mathcal{F}$, which gives rise to $\pi_{1*}\pi_2^*\mathcal{F} \to \pi_{1*}\pi_1^*\mathcal{F} \to \mathcal{F}$ by pre-composing with the trace map, and taking sections over X, we have

$$T_Q/U_Q : H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F})$$

for $\mathcal{F} = \omega^{\otimes k}$.

From the 'moduli -theoretic' viewpoint, these can be sees as:

$$f|(T_Q/U_Q)(E,P) = 1/p\sum f(E/D,(P+D)/D)$$

where the sum ranges over the $p+1$ finite flat subgroup schemes D of $E[p]$ of order p, while it ranges over all such D's which has only trivial intersection with the subgroup generated by P.

We would like to do this over the maximal unramified extension $L = W(\overline{\mathbf{F}}_p)[1/p]$ of $\mathbf{Q}_p$. Let O denote $W(\overline{\mathbf{F}}_p)$.

$X/\mathbb{C}$ has an integral model $\mathcal{X}$ over $\mathbf{Z}[1/N]$; it is smooth and geometrically irreducible such that

$$\mathcal{X}\times_{\mathbf{Z}[1/N]}\mathbb{C} \simeq X.$$

Let $\mathcal{X}_O$ denote $\mathcal{X}\times_{\mathbf{Z}[1/N]} O$. DRAW picture of the Raynaud generic fibre $\mathcal{X}_O[1/p]$ (this is isomorphic to the rigid analytic space of the Zariski generic fibre in the sense of Tate by the properness of $\mathcal{X}_O$):

Raynaud also defines a map of sites: $\mathrm{sp} : \mathcal{X}_O[1/p] \to \overline{\mathcal{X}}_O := \mathcal{X}_O \times_O \overline{\mathbf{F}}_p$. It is a result of Igusa that there are only finitely many supersingular points in $\overline{\mathcal{X}}_O$. We know that the completed local ring

$$O^{\wedge}_{\mathcal{X}_O,\nu} \simeq O[[\xi_\nu]]$$

for a supper-singular point ν in $\overline{\mathcal{X}}_O$ for a choice of parameter ξ_ν. The Raynaud generic fibre of $O^{\wedge}_{\mathcal{X}_O,\nu}$ is nothing other than $\mathrm{sp}^{-1}(\nu)$, and ξ_ν define a local coordinate of points in the generic fine which specialise to ν in the fibre.

Let $r \in p^{\mathbf{Q}}$ such that $1/p = p^{-1} < r \leq 1 = p^0$. For every supsersingular point ν in $\overline{\mathcal{X}}_O$, let

$$\mathrm{sp}^{-1}(\nu)_{<r}$$

denote the admissible open subset of points P in $\mathrm{sp}^{-1}(\nu)$ such that $(1/p <)|\xi(P)| < r$; and define

$$\mathcal{X}_O[1/p]_{\geq r} = \mathcal{X}_O[1/p] - \bigcup_{\nu} \mathrm{sp}^{-1}(\nu)_{<r}.$$

Define an element of $H^0(\mathcal{X}_O[1/p]_{\geq r}, \omega^{\otimes k})$ to be an r-overconvergent modular form of weight k and level $\Gamma_1(N)$ (defined over $L = O[1/p]$). An overconvergent modular form is a r-overconvergent modular form for some $r < 1$. When $r = 1$, the sections are called convergent modular forms 'of level prime to p' and are related to 'Katz p-adic modular forms'.

There are degeneracy morphisms

$$\pi_\nu : \mathcal{X}_{\nu,O} \longrightarrow \mathcal{X}_{\nu-1,O} \longrightarrow \cdots \longrightarrow \mathcal{X}_{1,O} \longrightarrow \mathcal{X}_{\mathrm{I},O} \longrightarrow \mathcal{X}_O$$

where $\mathcal{X}_t$ dente the O-module of the compactified modular curve of level $\Gamma_1(Np^t)$, $\mathcal{X}_{\mathrm{I},O}$ is of Iwahori level, and the map $\mathcal{X}_{t,O} \to \mathcal{X}_{\mathrm{I},O}$ is defined by sending (E, P) (where P is a 'point' of E order p^t) to $(E/\langle pP\rangle, \langle P/pP\rangle)$. The premise by π_ν of $\mathcal{X}_O[1/p]_{\geq r}$ has two components, and define $X_{\nu,O}[1/p]_{\geq r}$ to be the one which contains the cusp ∞ (i.e. containing th multiplicative ordinary locus).

For a supersingualr point ν of $\overline{\mathcal{X}}_{\mathrm{I},O}$, $O^{\wedge}_{\mathcal{X}_{\mathrm{I},O},\nu} \simeq O[[\xi, \xi^\vee]]/(\xi\xi^\vee - p)$, and its Raynaud generic fibre, $\mathrm{sp}^{-1}(\nu)$, is an annuli $\mathrm{Sp}\, L\langle \xi, p/\xi^\vee\rangle = \mathrm{Sp}\, L\langle \xi, \xi^\vee\rangle/(\xi\xi^\vee - p)$ whose $\overline{L}$-points is $\{z \in \overline{L} \mid 1/p = |p| \leq |z| \leq 1\}$. ELABORATE! DRAW A PICTURE

An overconvergent modular form of weight k and level $\Gamma_1(Np^\nu)$ is an element of $H^0(X_{\nu,O}[1/p]_{\geq r}, \omega^{\otimes k})$.

Remark. There is an entirely different way of defining 'overconvergent at Iwahori level' using Oort-Tate/Raynaud classification of finite flat group schemes. This is more natural.

A non-cuspidal point ξ of $\mathcal{X}_{\mathrm{I},O}[1/p]$ correspond to an (isomorphism class of) elliptic curve E which comes equipped with a finite flat (isotropic, i.e., the Weil paring $E[p] \simeq E[p]^\vee$ sends C isomorphically to $(E[p]/C)^\vee$) subgroup scheme $C \subset E[p]$ of order p over the ring R of integers of a finite extension of L. Oort-Tate carefully studies such C, and shows that it is of the form $\mathrm{Spec}\, R[t]/(t^p - \gamma t)$ while its Cartier dual $C^\vee$ is of the form $\mathrm{Spec}\, R[t^\vee]/((t^\vee)^p - \gamma^\vee t^\vee)$ for some $\gamma, \gamma^\vee$ in R such that $\gamma\gamma^\vee = p$. Now define $\deg(\xi)$ to be $1 - \mathrm{val}(\gamma) = \mathrm{val}(\gamma^\vee)$, where val is a normalised valuation such that $\mathrm{val}(p) = 1$. The degree tells us $\deg(\xi) = 0$ (resp. $= 1$) if and only if C is multiplicative (resp. étale) *and* E reduces to an ordinary elliptic curve in characteristic p; and $0 < \deg(C) < 1$ if and only if E reduces to a supersingular elliptic curve. A remarkable observation one can make is, knowing deg, one more or less knows p-adic geometry of modular curves (well, at least for applications people in the trade have in mind).

Proposition 12 *A classical modular form is overconvergent.*

Proof. This follows by definition. □

The converse, however, does not necessarily hold. However, arguably one of the most important results in p-adic theory of modular form is:

Theorem 13 *Let f be an overconvergent modular form of weight $k \geq 2$ and of level $\Gamma_1(Np^\nu)$. Suppose that $U_p f = \lambda f$ and λ is a non-zero element of L. If* $\mathrm{val}(\lambda) < k - 1$*, then f is classical.*

This is a really hard theorem. It is perhaps difficult to pin down a criterion for 'being classical' when $k = 1$ without reference to its associated Galois representation (which exists).

It is a result of Kisin and Colmez that the Galois representation $\rho = \rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(O)$ associated to f as in the theorem is triangulate at p, i.e., its associated (φ, Γ)-module is reducible. When $\mathrm{val}(\lambda) = 0$, the associated representation ρ is ordinary, i.e., reducible (and has a one-dimensional sub, or a quotient, that is unramified; this can be arranged by twisting ρ by a character) at p, which is an old result of Wiles. Note that, in both case, their 'Fontaine modules' are reducible. The case for forms with $\lambda = 0$, the infinite slope case is completely mysterious; they correspond to cuspidal automorphic representations which are supersupidal at p.

Recall that a point φ corresponds to a $\mathbb{C}_p^\times$-valued character of $\mathbf{Z}_p^\times \times (\mathbf{Z}/N\mathbf{Z})^\times \simeq (\mathbf{Z}/pN\mathbf{Z})^\times \simeq (1 + p\mathbf{Z}_p)^\times$; if φ lies, in particular, in $\mathcal{W}_\chi(\mathbb{C}_p)$, then φ, when restricted to $(\mathbf{Z}/Np\mathbf{Z})^\times$, equals χ.

We shall be interested in characters which lie in $\mathbf{B}_{<\lambda}$ for $\lambda = p^{1-1/(p-1)}$. Coleman call them *accessible* with no motivation whatsoever for the terminology, but my educated guess would be as follows: for $N \in \mathbf{B}_{<\lambda}$ and $\xi = [\xi]\{\xi\} \in \mathbf{Z}_p^\times \simeq (\mathbf{Z}/p\mathbf{Z})^\times \times (1 + p\mathbf{Z}_p)^\times$, it makes sense ('accessible') to define

$$\{\xi\}^N := \exp(N \log\{\xi\}).$$

As $|N| < \lambda$, $|N\log\{\xi\}| = |N||\log(1 + (\{\xi\} - 1))| < p^{1-1/(p-1)}p^{-1} = p^{-1/(p-1)}$ (note that by definiton, $|\{\xi\}| \leq p^{-1} < p^{-1/(p-1)}$, and hence this is convergent (Recall from p-adic analysis that $\exp X = \sum_{N=0}^\infty X^N/N!$ is convergent if $|X| < p^{-1/(p-1)}$ (See Washington p.49) and $\log(1 + X) = \sum_{N=1}^\infty (-1)^{N+1} X^N/N$ satisfies $|\log(1 + X)| \leq |X|$ if $|X| \leq p^{-1/(p-1)}$ (See Washington Lemma 5.5, p.51)). So accessible characters are characters which sends $\{\xi\} \in (1 + p\mathbf{Z}_p)^\times$ to $\{\xi\}^N$ for $N \in \mathbf{B}_{<\lambda}$.

We shall define overconvergent modular forms of level $\Gamma_1(Np)$ and of weight $\varphi \in \mathcal{W}_{\chi,<\lambda}$.

Definition. Define ζ (by slight abuse of notation, but I see no harm doing so) on $\mathcal{W}$ as a 'family' of p-adic L-functions defined such that, if φ is a (non-trivial) accessible character, i.e., a character which sends ξ to $\{\xi\}^N \chi(\xi)$ for N in $\mathbf{B}_{<\lambda}$ and χ is a charcter of finite order, then $\zeta(\varphi) = L_p(1 - N, \chi)$. More precisely, it is defined (in Lang's Cyclotomic Fields Chapter 4, Sec. 3. I shall elaborate this a little in Appendix) to be

$$\zeta(\varphi) = (\varphi(\rho) - 1)^{-1} \int_{\mathbf{Z}_p^\times} \varphi(M) M^{-1}\, d\mu(M)$$

for *any* $\rho \in \mathbf{Z}_p^\times$ such that $\varphi(\rho)$ does not equal 1, in the notation of Lang (in particular, $\mu = \mu_{1,\rho}$ is a measure on $\mathbf{Z}_p^\times$ defined in Chapter 2; see my Appendix).

Definition. For φ in $\mathcal{W}$, let $G_\varphi = G_\varphi(q) = \zeta(\varphi)/2 + \sum_{N\geq 1} \sigma_\varphi(N) q^N$, where $\sigma_\varphi(N)$ denotes the sum, over all integers $0 < M$ which are coprime to p and which divide N, of $\varphi(M)/M$.

Proposition 14 *Suppose that φ is a character of $\mathbf{Z}_p^\times$ sending ξ to $\chi(\xi)\{\xi\}^k$ for $k \in \mathbf{Z}$ and a character χ of $\mathbf{Z}_p^\times$ of conductor Np^λ (where N is prime to p and $\lambda \geq 0$). Then G_φ is the q-expansion of an overconvergent modular form of weight k and level $\Gamma_1(Np^\nu)$ where $\nu = \mathrm{LCM}\{1, \lambda\}$ with character $\chi/[\xi]^k$ on $(\mathbf{Z}/Np^\nu\mathbf{Z})^\times$. If k furthermore satisfies ≥ 1, G_φ is the q-expansion of a classical modular form.*

Definition. Let $E_\varphi = E_\varphi(q) = (\zeta(\varphi)/2)^{-1} G_\varphi$. In particular, let E denote E_φ for the character φ sending ξ to $\{\xi\}$; it is overconvergent of weight 1, and of level $\Gamma_1(p)$ with character $\xi \mapsto [\xi]^{-1}$.

Definition. A series $F = \sum_N c_N q^N \in \mathbb{C}_p[[q]]$ is called (the q-expansion of) an overconvergent modular form of weight $\varphi \in \mathcal{W}_{\chi,<\lambda}$ if F/E^φ is the q-expansion of an overconvergent function in $O(\mathcal{X}_{1,O}[1/p]_{\geq r})$ for $p^{-1/(p+1)} < r < 1$.

For an admissible open subset U of $\mathcal{W}_{\chi,<\lambda}$, a formal q-expansion $F = \sum_N C_N q^N \in O(U)[[q]]$ is called (the q-expansion of) a family of overconvergent modular form if for any φ which lies in $U(\mathbb{C}_p)$, $\varphi^* F/E^\varphi \in \mathbb{C}_p[[q]]$ is the q-expansion of an overconvergent function in $O(\mathcal{X}_{1,O}[1/p]_{\geq r})$ for $p^{-1/(p+1)} < r < 1$.

These definitions, however, makes it difficult to define the U_p-operator! Suppose that f is an r-overocnvergent modular form of weight k and level $\Gamma_1(Np)$ for $p^{-1/(p+1)} < r < 1$, and suppose that it does not vanish on the domain $\mathcal{X}_{1,O}[1/p]_{\geq r}$. Then one can check that $U = U_p$ may be defined as

$$M_k^{\geq r}(\Gamma_1(Np); O) \xrightarrow{f^{-1}} O(\mathcal{X}_{1,O}[1/p]_{\geq r}) \xrightarrow{U\circ e} O(\mathcal{X}_{1,O}[1/p]_{\geq r^{1/p}}) \hookrightarrow O(\mathcal{X}_{1,O}[1/p]_{\geq r}) \xrightarrow{f} M_k^{\geq r}(\Gamma_1(Np); O)$$

where U is the standard 'Hecke correspondence' defined by $(f|U)(E,P) = 1/p \sum f(E/D, P/D)$ where the sum ranging over $D \subset E[p]$ which intersect only trivially with the subgroup generated by P, and $e = f(q)/f(q^p)$. The following lemma ascertains that there is a substitute 'in families' of the function e:

Lemma 15 *There exists a rigid analytic function e on $\sum_{r\in(p^{-1/(p+1)},1]} \mathcal{X}_{1,O}[1/p]_{\geq r}$ whose q-expansion is exactly $E(q)/E(q^p)$. For any $\epsilon \in \mathbf{R}$ such that $|p| < \epsilon$, there exists $r \in (p^{-1/(p+1)}, 1]$ such that e is defined on $\mathcal{X}_{1,O}[1/p]_{\geq r}$ and $|e-1| \leq \epsilon$.*

Applying the lemma to $N \in \mathbf{B}_{<\lambda}$ and $\epsilon = p^{-1/(p-1)}$, there exists $r = r_N$ such that $|e-1| < p^{-1} < p^{-1/(p-1)}$. The importance of these choices is that $e^N = \exp(N\log(e)) = \exp(N\log(1+(e-1)))$, so if $|e-1| < p^{-1/(p-1)}$ does hold, $|\log(1+(e-1))| \leq |e-1|$ whence

$$|N\log(1+(e-1))| \leq |N||e-1| < p^{1-1/(p-1)}p^{-1} = p^{-1/(p-1)}$$

and e^N converges!.

So, for an accessible character φ which sends ξ to $\chi(\xi)\{\xi\}^N$ for $N \in \mathbf{B}_{<\lambda}$ for a character χ of $(\mathbf{Z}/p\mathbf{Z})^\times$, define U to be

$$M_\varphi^{\geq r}(\Gamma_1(Np); O(\mathcal{W}_{\chi,<\lambda})) \xrightarrow{E^{-N}} O(\mathcal{X}_{1,O}[1/p]_{\geq r}) \xrightarrow{U\circ e^N} O(\mathcal{X}_{1,O}[1/p]_{\geq r^{1/p}}) \hookrightarrow O(\mathcal{X}_{1,O}[1/p]_{\geq r}) \xrightarrow{E^N} M_k^{\geq r}(\Gamma_1(Np); O(\mathcal{W}_{\chi,<\lambda})).$$

It should be clear by now that the main thrust of Colema's idea is that, as we are 'lucky' enough to find E_φ and e, one can 'twist back and forth' to reduce problems to functions on $\mathcal{X}_{1,O}[1/p]_{\geq r}$. On the other hand, r does not remain constant as e goes deeper into $\mathbf{B}$ ('higher p-power levels' which lead indeed to less overconvergence allowed). Keeping track of these parameters carefully, Coleman constructs families of overconvergent *eigenforms*. Perhaps it is a useful mental exercise to ask oneself: in constructing families of modular forms of finite slope, what is it that one should 'interpolate'?, or in exactly what sense, should one 'interpolate'? Since slopes are no longer fixed zero as in the ordinary case, this question is legitimately tricky. Coleman's answer is to interpolate the 'Hecke polynomials' of U, or more precisely, the characteristic power series $\det(1-TU)$ acting on 'spaces of modular forms'. Lucky, the complete-continuity of U singles out right pieces from families of overconvergent modular forms to makes sense of U-eigenforms, which Coleman achieve by making appeal to Serre's theory of Banach spaces.

It is indeed possible to bypass 'Coleman's trick'. One instead aims to construct automorphic bundles of general weight and families by extending the 'torsor' structure of the Igusa covering to tubular neighbourhoods of the ordinary locus; this is achieved by the Hodge-Tate map, which relates subgroups 'C' to the ω of $E[p]$, and hence to the 'automorphic bundle' ω. See Andreatta-Iovita-Stevens/Pilloni.

An analogue of Hida's Λ-adic forms is a construction of Coleman-Mazur eigencurve: there exists a rigid analytic L-variety

$$f : \mathcal{C} \to \mathcal{W} = \coprod_\chi \mathcal{W}_\chi$$

where χ ranges over $(\mathbf{Z}/pN\mathbf{Z})^\times$.

For every $\mathbb{C}_p$-point φ of $\mathcal{W}$, the fibre $f_\varphi \in \mathcal{C}$ defines an overconvergent modular eigenform of weight κ and level Np^ν for some ν depending on φ.

Fix a non-zero rational number ρ. For any $\varphi \in \mathcal{W}_{\chi,<\lambda}(\mathbb{C}_p)$, there exists a ball $\mathbf{B}_r \subset \mathcal{W}_\chi$ of radius r centred at φ and an admissible open subset $U_{\rho,r} \subset \mathcal{C}$ mapping by f to $\mathbf{B}_r \hookrightarrow \mathcal{W}_\chi$ such that, as a point ξ of $\mathbf{B}_r$, corresponding to the character sending u (thought of as a parameter in $\mathcal{W}_\chi$) to $(1+p)^k\zeta$ (for a primitive p^ν-th root of unity ζ), ranges over the integers $k \geq 2$ satisfying $k > \rho + 1$, it gives an bijection between their fibres $\{f_\xi\}_\xi$ in $U_{\rho,r}$ and the classical cuspidal modular eigenform of weight k, level $Np^{\nu+1}$

(where level 'Np' is accounted for by χ), and of slope ρ. In other words, locally, $\mathcal{C}$ interpolates classical modular eigenforms of finite slope.

Note that 'overconvergent modular forms of weight φ' is defined to be a function on $\mathcal{X}_{\nu,O}[1/p]_{\geq r}$ multiplied by the Eisenstein series E_φ of weight φ, and there is no 'funny' twist in making sense of weights (in Hida theory, $\langle\ \rangle$ is used to define a map down to the weight space, while in Coleman theory, the corresponding map, therefore, is basically twist by weight characters themselves).

6 Appendix: p-adic L-functions

I shall elaborate a little about Coleman's ζ, by following Lang's Cyclotomic fields (see also Mazur's unpublished notes 'Analyse p-adique'; perhaps, to hear it from the horse's mouth, Iwasawa's PUP lecture notes on p-adic L-functions would also help), which is necessary in writing down a concrete/explicit example of p-adic families (Eisenstein families). OTOH, this would inevitably involve 'p-adic L-functions', a deep subject on its own, which I was keen to avoid in the lectures altogether. Perhaps the best thing to do is to make an attempt to read one of the references I've just given, but the following exposition (following Lang, because Coleman does) has the benefit of looking much less terrifying.

Let $X = \{X_N; \pi_{N+1} : X_{N+1} \twoheadrightarrow X_N\}_N$ be an injective system/limit of finite sets X_N's. By a function f on X, I shall mean a system of functions $f = \{f_N\}_N$ where each f_N is defined on X_N and the functions are compatible, in the sense that, for ξ in X_N, $f_N(\xi) = \sum f_{N+1}(\zeta)$ holds, where the sum ranges over ζ in X_{N+1} in the pre-image of ξ by π_{N+1}. Observe that is f factors though X_N, it factors though X_{N^+} for $N^+ \geq N$ 'by projection'.

Given a locally constant function f on X (i.e. a function which factors through X_N for some N) and a function $\mu = \{\mu_N\}_N$ on X, one defines

$$\int_X f d\mu = \sum f(\xi)\mu_N(\xi)$$

where the sum ranges over ξ in X_N, and $d\mu$ or the system $\mu = \{\mu_N\}_N$ is called a distribution on X. By the compatibility of f, the $\sum_{X_N} f(\xi)\mu_N(\xi) = \sum_{X_{N^+}} f(\xi)\mu_{N^+}(\xi)$.

We say that μ is bounded when $|\mu_N(\xi)|$ is bounded for every ξ and N. When μ is indeed bounded, for a continuous function f on X, f is approximated uniformly by locally constant functions, i.e., $||f - f_N|| \to 0$ as N tends to ∞ (where $||\ ||$ denotes the sup norm), hence $||f_M - f_N|| \to 0$ a $M, N \to \infty$. In particular, $\int f_N d\mu$ converges, and let

$$\int f d\mu = \lim_N \int f_N d\mu.$$

6.1 An example of distributions: Bernoulli distribution

Let the polynomial $B_\lambda(X)$ in X be defined by $te^{tX}/e^t - 1 = \sum_{\lambda=0}^\infty B_\lambda(X)t^\lambda/\lambda!$. When $X = 0$, it gives rise to the Bernoulli number B_λ.

Proposition 16

- *The function sending ξ to $N^{\lambda-1}B_\lambda(\langle\xi\rangle)$ defines a distribution on $\{N^{-1}\mathbf{Z}/\mathbf{Z}\}_N$, where for any ρ in $\mathbf{R}/\mathbf{Z}$, $\langle\rho\rangle$ denote the smallest non-negative real number in the class of ρ in $\mathbf{R}/\mathbf{Z}$;*
- *the function μ_λ^N sending ξ to $N^{\lambda-1}\lambda^{-1}B_\lambda(\langle\xi/N\rangle)$ defines a distribution μ_λ on $\{\mathbf{Z}/N\mathbf{Z}\}_N$.*

Proposition 17 *For the distribution μ_λ on $\{\mathbf{Z}/p^\nu\mathbf{Z}\}_\nu$,*

$$B_\lambda/\lambda = \int_{\mathbf{Z}_p} d\mu_\lambda$$

for $\mu = \mu_\lambda$.

Proof. 1 is locally constant function (which is thought of as factoring through $\mathbf{Z}/p\mathbf{Z}$) and, by definition, the RHS is $\sum 1\mu_\lambda(\xi) = \sum p^{\lambda-1}\lambda^{-1}B_\lambda(\langle\xi/p\rangle) = \lambda^{-1}\sum p^{\lambda-1}B_\lambda(\langle\xi/p\rangle) = \lambda^{-1}B_\lambda(0) = \lambda^{-1}B_\lambda$. The non-trivial equality follows from

$$B_\lambda(X) = p^{\lambda-1}\sum_{\nu=1}^{p-1} B_\lambda((X+\nu)/p)$$

which can be proved by unravelling the definition. Otherwise, see B3 in Section 2 in Chapter 2 of Lang. □

More generally, for a positive integer M, let χ be a function on $\mathbf{Z}/M\mathbf{Z}$, and define the generalised 'Bernoulli polynomial' $B_{\lambda,\chi}(X)$ by $\sum_{\nu=1}^{M-1}\chi(\nu)te^{t(\nu+X)}/e^{tM}-1 = \sum_{\lambda=0}^{\infty}B_{\lambda,\chi}(X)t^\lambda/\lambda!$

If χ is a Dirichlet character on $\mathbf{Z}/M\mathbf{Z}$ extended from $(\mathbf{Z}/M\mathbf{Z})^\times$ by zeros, $B_{\lambda,\chi}(0)$ *is* Leopoldt's Bernoulli number $B_{\lambda,\chi}$.

Note that, while μ_λ is a distribution, it is not in general a measure. However,

Proposition 18 *For a rational number ρ prime to M (i.e. M is prime to both the numerator and the denominator), the function $\mu^N_{\lambda,\rho}$ which sends ξ to $\mu^N_\lambda(\xi) - \rho^\lambda\mu^N_\lambda(\rho^{-1}\xi)$ defines a measure $\mu_{\lambda,\rho}$ on $\{\mathbf{Z}/N\mathbf{Z}\}_N$.*

We shall now let $N = p^\nu$, in which case $X = \mathbf{Z}_p = \{\mathbf{Z}/p^\nu\mathbf{Z}\}_\nu$. Suppose also M is a p-power, so that χ is thought of as a locally constant function on $\mathbf{Z}_p$ factoring through $\mathbf{Z}/M\mathbf{Z}$.

Proposition 19 *If χ is locally constant on $\mathbf{Z}_p$ (so μ doesn't have to be bounded to integrate it!)*

$$B_{\lambda,\chi}/\lambda = \int_{\mathbf{Z}_p}\chi d\mu$$

for $\mu = \mu_\lambda$. In other words,

$$B_{\lambda,\chi} = M^{\lambda-1}\sum_{\nu=1}^{M-1}\chi(\nu)B_\lambda(\langle\nu/M\rangle).$$

Proof. Similar to the case when χ is '1' proved above.□

Proposition 20 *If ρ is an element of $\mathbf{Z}_p^\times$,*

$$\mu_{\lambda,\rho}(\xi) = \xi^{\lambda-1}\mu_{1,\rho}(\xi).$$

Theorem 21 *If ρ is an element of $\mathbf{Z}_p^\times$ and λ is a positive integer such that $\rho^\lambda \neq 1$,*

$$B_\lambda/\lambda = (1-\rho^\lambda)^{-1}\int_{\mathbf{Z}_p}\xi^{\lambda-1}d\mu(\xi)$$

where $\mu = \mu_{1,\rho}$.

Proof. See Theorem 2.3 in Section 2, Chapter 2, Lang. □

As corollaries, one can prove 'Kummer congruences' and 'Von Staudt congruence'.

Theorem 22 *If χ is a character of finite order on $\mathbf{Z}_p^\times$,*

$$B_{\lambda,\chi}/\lambda = (1-\chi(\rho)\rho^\lambda)^{-1}\int_{\mathbf{Z}_p^\times}\chi(\xi)\xi^{\lambda-1}d\mu(\xi)$$

where $\mu = \mu_{1,\rho}$.

Proof. See Theorem 2.4 in Section 2, Chapter 2, Lang. □

Recall that Coleman defines $\zeta(\varphi)$, for φ in $\mathcal{W}(\mathbb{C}_p)$ to be $(\varphi(\rho)-1)^{-1}\int_{\mathbf{Z}_p^\times}\varphi(\xi)\xi^{-1}\,d\mu(\xi)$ for $\mu=\mu_{1,\rho}$ such that the denominator is well-defined. Hence φ is the character of $\mathbf{Z}_p^\times$ sending ξ to $\{\xi\}^N\chi(\xi)$ for a character χ of finite order, it gives rise to the Kubota-Leopoldt p-adic L-function $L_p(1-N,\chi)$. If N is furthermore a positive integer, $\zeta(\varphi)=L_p(1-N,\chi)=-B_{N,\chi/[\ ]^N}/N$ (the generalised Bernoulli number with character $\chi/[\ ]^N$ (since $\varphi(\xi)=\chi(\xi)\{\xi\}^N=\chi(\xi)(\xi/[\xi])^N=(\chi(\xi)/[\xi]^N)\{\xi\}^N$).

It is a reasonable guess now that there should be a p-adic L-functions defined over Coleman p-adic families of p-adic overconvergent eigenforms of a (fixed) finite slope, and by extension to the Coleman-Mazur/Buzzard eigencurve. See papers of Bellaiche, Emerton, Stevens, R.Pollack, Loeffler-Zerbes and many others I have failed to mention due to my ignorance (please let me know before I embarrass myself), if you are interested.