Serre's conjecture about weights of mod p Hilbert modular forms

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Before I begin

Everything I say today is joint work with F. Diamond (DS). Some part of it is joint with P. Kassaei (DKS).

My notes below contain a lot more details than what I intend to say.

What is this talk about?

This talk is about unravelling how

- mod p algebraic geometry
- and mod p representation theory

of GL_2 (over a totally real field) are related in the context of mod p theory of automorphic forms.

Some of what we've done below would undoubtedly be useful in formulating a 'mod p Langlands philosophy'.

Introduction

Let $p > 2$ be a rational prime. Let

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous and odd (if c is the complex conjugation in the decomposition subgroup $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ at infinity, then $\det\overline{\rho}(c) = -1$).

Old conjecture

J.-P. Serre (1987) defined/specified

- $k(\overline{\rho}) \geq 2$
- $N(\overline{\rho}) \geq 1$, the Artin conductor prime to p

and conjectured that there should be a cuspidal modular eigenform F of weight $k(\overline{\rho})$ and level $N(\overline{\rho})$ such that (for a choice of $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$),

$$F \rightsquigarrow \overline{\rho}_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_F} \mathrm{GL}_2(\overline{\mathbb{Z}}_p) \twoheadrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is isomorphic to $\overline{\rho}$.

Not so old theorem

Theorem (Khare-Wintenberger 2009)

Serre's conjecture, as stated above, holds.

Example of Serre's $k(\overline{\rho})$

Suppose that $\overline{\rho}$ is reducible at p. In fact, suppose that $\overline{\rho}$ is tamely ramified at p. In this case,

$$\overline{\rho}|_I \simeq \begin{pmatrix} \epsilon^{k_1} & 0 \\ 0 & \epsilon^{k_2} \end{pmatrix}$$

where ϵ is the mod p cyclotomic character $\epsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{F}}_p^\times$. WLOG, we may assume $0 \leq k_1 \leq k_2 \leq p-2$.

Serre's recipe says

$$k(\overline{\rho}) = 1 + pk_1 + k_2$$

if (k_1, k_2) is not $(0,0)$; while

$$k(\overline{\rho}) = p$$

if $(k_1, k_2) = (0,0)$, i.e., $\overline{\rho}$ is unramified at p.

Geometric mod p modular forms

Serre realised that it was possible to follow Katz to define modularity of $\overline{\rho}$ differently: $\overline{\rho}$ is modular of weight k and of level N if it arises from an eigenform in the sections

$$H^0(X_{\Gamma_1(N)} \times \overline{\mathbb{F}}_p, \omega^k)$$

with $k = k(\overline{\rho})$ and $N = N(\overline{\rho})$, where

- $X_{\Gamma_1(N)}$ is the compactified modular curve of level $\Gamma_1(N)$ over $\mathbb{Z}_p$,
- $\omega = s_*\Omega_{E/Y_{\Gamma_1(N)}}$ over $Y_{\Gamma_1(N)} \rightsquigarrow \omega$ over $X_{\Gamma_1(N)}$.

Replace ω by $\omega(-Z_{\Gamma_1(N)})$ for cusp forms.

What does this change of perspective entail?

- there are modular forms that may exist only over $\overline{\mathbb{F}}_p$ (and not lift over to $\overline{\mathbb{Q}}_p$):

$$\to H^0(X_\Gamma, \omega^k(-Z_\Gamma)) \to H^0(\overline{X}_\Gamma, \overline{\omega}^k(-\overline{Z}_\Gamma)) \to$$

  is not necessarily surjective when $k = 1$,
- for a weight $k = 1$ eingenform f in $H^0(\overline{X}_\Gamma, \overline{\omega}(-Z_\Gamma))$

$$f \rightsquigarrow g = fh \rightsquigarrow \text{an eigen } G \in H^0(X_\Gamma, \omega^p(-Z_\Gamma)) \rightsquigarrow \rho_G \rightsquigarrow \overline{\rho}_G =: \overline{\rho}_f$$

- the weight recipe needs to be modified accordingly– when $\overline{\rho}$ is unramified at p, $k(\overline{\rho}) = 1$ rather than p. The new $k(\overline{\rho})$ is minimal/smallest possible (Edixhoven 1992).

  [If $\overline{\rho}$ is modular, $k(\overline{\rho})$ is exactly the weight filtration of $f(\overline{\rho})$]

Mod p Langlands correspondence

A neat consequence (still assuming $p > 2$):

"Theorem"
There exists a 'correspondence' between

- eigenforms in $H^0(X_{\Gamma_1(N)} \times \overline{\mathbb{F}}_p, \omega^k)$ of 'minimal' weight $k \geq 1$ and 'minimal' level N prime to p (with $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$),
- odd continuous representations $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ of conductor N prime to p such that $\overline{\rho}_p := \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ has a crystalline lift of 'minimal' HT weight $(k-1, 0)$.

This follows from work of Khare-Wintenberger, Kisin, Taylor, Emerton, Diamond, Wiles, Carayol, Ribet, Coleman-Voloch, Gross, Edixhoven...

Local-global compatibility

In some sense, local-global compatibility at p (resp. away from p) manifests itself as 'weight' $k(\overline{\rho})$ (resp. 'level' $N(\overline{\rho})$).

Serre's conjecture in the Hilbert case

Our motivation was, in some sense, to generalise the mod p Langlands correspondence above with $\mathbb{Q}$ replaced by a totally real field F– in particular local-global compatibility at p.

This was initiated by Buzzard-Diamond-Jarvis (2010) for 'regular weights $k \geq 2$'.

Suppose that p is inert in F (throughout my talk today).

In my forthcoming joint work with F. Diamond, we deal with the general ramified case (i.e. no assumption on p relative to F).

Fix $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{F}}_p$ and fix embeddings

$$\imath : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$$

and

$$\jmath : \overline{\mathbb{Q}} \to \mathbb{C}$$

once for all.

Let

$$\Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}).$$

By definition,

$$\imath \circ \Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$$

and

$$\jmath \circ \Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R}).$$

$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$

To understand geometry of the Shimura variety of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, it is necessary to work with the Shimura variety of

$$\begin{array}{ccc} G & \to & \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2 \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \to & \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \end{array}$$

(of level full congruence) and 'descend' to that of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, but I am going to use them interchangeably. So let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$$

[By abuse of notation, often use G to denote its model, e.g. $G(\mathbb{F}_p) = \mathrm{GL}_2(\mathscr{O}_F/p)$]

Models of HMFs

- $\Gamma \subset G(\mathbb{A}^\infty)$ maximal compact hyperspecial at p, which we always assume sufficiently small,
- (Raapoport/Deligne-Pappas) an integral $\mathbb{Z}_p$-model Y_Γ for

$$G(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R})^\Sigma \times G(\mathbb{A}^\infty)/\Gamma$$

- associated to $(k, \ell) \in \mathbb{Z}^{\imath \circ \Sigma} \times \mathbb{Z}^{\imath \circ \Sigma}$, we have the automorphic line bundle

$$\mathscr{A}_{(k,\ell)} = \bigotimes_{\tau \in \imath \circ \Sigma} \omega_\tau^{k_\tau} \otimes \delta_\tau^{\ell_\tau}$$

where

$$\omega = s_* \Omega_{A/Y_\Gamma} = \bigoplus_\tau \omega_\tau$$

and

$$\delta = \bigwedge^2_{\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}} R^1 s_* \Omega^\bullet_{A/Y_\Gamma} = \bigoplus_\tau \delta_\tau .$$

Mod p HMFs

The space of mod p Hilbert modular forms of weight (k, ℓ) are defined to be

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)})$$

where $\overline{Y}_\Gamma = Y_\Gamma \times \overline{\mathbb{F}}_p$.

Note that, since p is (in particular) unramified, we identify:

$$\imath \circ \Sigma \simeq \mathrm{Hom}_{\mathbb{F}_p}(\mathscr{O}_F/p, \overline{\mathbb{F}}_p).$$

The Frobenius Φ acts on Σ. If $|\mathscr{O}_F/p| = p^f$ and fix τ in Σ,

$$\Sigma = \{\tau, \Phi \circ \tau, \ldots, \Phi^{f-1} \circ \tau\} \simeq \mathbb{Z}/f\mathbb{Z}$$

$F = \mathbb{Q}$

Recall, when $F = \mathbb{Q}$ (and $p > 2$), that

- if a mod p cusp form f is not in the image of

  $$H^0(X_\Gamma, \omega^k(-Z_\Gamma)) \to H^0(\overline{X}_\Gamma, \overline{\omega}^k(-\overline{Z}_\Gamma)),$$

  then either (1) $k = 1$ or (2) $k \geq 12$, $p = 3$ and $N = 1$.

  [(Serre/Carayol) If it is about lifting eigenforms with characters intact, exclude f in (2) such that $\overline{\rho}_f$ is induced from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$.]

- $H^0(X_\Gamma, \omega^k(-Z_\Gamma))$ is 0 for negative k.

$F \neq \mathbb{Q}$

When $F \neq \mathbb{Q}$,

- every Hilbert modular form of weight (k, ℓ) over $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ has its weight paritious, i.e. $k_\tau + 2\ell_\tau$ is independent of τ.

  There are a lot more mod p Hilbert modular forms that are not in the image of

  $$H^0(Y_\Gamma, \mathscr{A}_{(k,\ell)}) \to H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)}).$$

- In stark contract to the case $F = \mathbb{Q}$, there are lots of mod p Hilbert modular forms of 'negative weights'.

Example ($\ell = 0$)

For every τ in $\imath \circ \Sigma$, let

$$
\begin{aligned}
& V = F_{A^\vee}^\vee : A^{(p)} = A \times_{\overline{Y}_\Gamma, \Phi} \overline{Y}_\Gamma \to A \\
\rightsquigarrow \quad & \omega_\tau = (s_* \Omega_{A/\overline{Y}_\Gamma})_\tau \xrightarrow{V^*} (\Phi^* s_* \Omega_{A/\overline{Y}_\Gamma})_\tau = \omega^p_{\Phi^{-1} \circ \tau} \\
\rightsquigarrow \quad & H_\tau \in H^0(Y_\Gamma \times \overline{\mathbb{F}}_p, \omega^p_{\Phi^{-1} \circ \tau} \otimes \omega_\tau^{-1})
\end{aligned}
$$

denote the partial Hasse invariant at τ of weight

$$h_\tau = (0, \ldots, 0, p, -1, 0, \ldots, 0) = p 1_{\Phi^{-1} \circ \tau} - 1_\tau$$

where p (resp. -1) stands at $\Phi^{-1} \circ \tau$ (resp. τ).

Automorphic Galois representations

Theorem (DS)

Let f be an element $H^0(Y_\Gamma \times \overline{\mathbb{F}}_p, \mathscr{A}_{(k,\ell)})$ and S be a finite set of finite places in F, containing all v dividing p and all v such that $\mathrm{GL}_2(\mathscr{O}_{F_v}) \not\subset \Gamma$.
Suppose that

$$T_v f = \alpha_v f$$

and

$$S_v f = \beta_v f$$

for all v not in S. Then there exists a continuous representation

$$\overline{\rho}_f : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is unramified outside S and the characteristic polynomial in X of $\rho_f(\mathrm{Frob}_v)$ is

$$X^2 - \alpha_v X + \beta_v \mathrm{N}_{F/\mathbb{Q}}(v).$$

Remarks

The novelty of our theorem is that (k, ℓ) does not have to satisfy the parity condition that $k_\tau + 2\ell_\tau$ is independent of τ in Σ. The parity case is known by Emerton-Reduzzi-Xiao and Goldring-Koskivirta.

How do we deal with HMFs of non-paritous weight? We lift mod p HMFs of parallel weight but of level $\Gamma \cap \Gamma_1(p)$.

Conjecture (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be totally odd, continuous and irreducible. Then $\overline{\rho}$ is modular in the sense above.

In preparation of the DS conjecture

Definition (the Diamond-Kassaei minimal cone)

$$\Xi = \{k = \sum_{\tau \in \imath \circ \Sigma} k_\tau \tau \mid pk_\tau \geq k_{\Phi^{-1} \circ \tau}\} \subset \mathbb{Z}^{\imath \circ \Sigma}$$

and

$$\Xi^+ = \{k \in \Xi \mid k_\tau \geq 1\}.$$

Definition

$$k \succeq k'$$

if $k - k'$ is a non-negative integer linear combination of the weights h_τ of the partial Hasse invariants.

Conjecture

Conjecture (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\iota\circ\Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ^+ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \succeq k(\overline{\rho}, \ell)$
- if $k \in \Xi^+$, then $k \succeq k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_p)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

$\rightsquigarrow$ a mod p Langlands correspondence for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$.

Example when $F = \mathbb{Q}$ and $\ell = 0$

$$\Xi = \{k \geq 0\},$$
$$\Xi^+ = \{k \geq 1\},$$
$$k \succeq k' \text{ if } k - k' = (p-1)n \geq 0.$$

There exists $k(\overline{\rho}) \geq 1$ such that the following are equivalent:

- $\overline{\rho}$ is modular of weight k,
- $k \succeq k(\overline{\rho})$,
- $\overline{\rho}_p$ has a crystalline lift of weight $(k-1, 0)$,

for every $k \geq 1$.

$\rightsquigarrow$ $k(\overline{\rho})$ is the smallest possible weight for which $\overline{\rho}$ is modular.

How is our conjecture related to the Buzzard-Diamond-Jarvis conjecture?

Serre weights

It is well-known that an irreducible representation theory of $G(\mathbb{F}_p)$ is of the form

$$V_{k,\ell} = \bigotimes_{\tau \in \Sigma} \det{}^{\ell_\tau} \otimes \mathrm{Sym}^{k_\tau - 2} \left(V \otimes_{\mathscr{O}_F, \tau} \overline{\mathbb{F}}_p \right)$$

where V is the standard representation of $G(\mathbb{F}_p)$ on two copies of $\mathscr{O}_F / p$ and where

$$0 \leq \ell_\tau \leq p - 1,$$

but not all ℓ_τ are simultaneously $p - 1$ and

$$0 \leq k_\tau - 2 \leq p - 1.$$

We often call such a representation a Serre weight.

Algebraic modularity (BDJ)

Definition

We say that $\overline{\rho}$ is algebraic modular of Serre weight V, we mean that there exist

- a quaternion algebra D over F, split at exactly one infinite place and all place above p: $(D \otimes_F F_p)^\times \simeq G(\mathbb{Q}_p)$,
- a sufficiently small open compact subgroup $\Gamma \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ containing $G(\mathbb{Z}_p)$

such that $\overline{\rho}$ is a $\overline{\mathbb{F}}_p[\mathrm{Gal}(\overline{F}/F)]$-subquotient of

$$(\mathrm{Pic}^0 X^D_{\ker(\Gamma \to G(\mathbb{F}_p))}[p](\overline{F}) \otimes V)^{G(\mathbb{F}_p)}$$

where $G(\mathbb{F}_p) = \mathrm{GL}_2(\mathscr{O}_F/p)$ acts diagonally over $\otimes$ and $\mathrm{Gal}(\overline{F}/F)$ acts trivially on V.

Algebraic modular = Geometric modular?

Conjecture (DS)

Let $(k, \ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ and $k_\tau \geq 2$ for every τ in Σ. If $\overline{\rho}$ is algebraic modular of weight (k, ℓ), i.e., of Serre weight

$$V_{k,1-k-\ell} = \bigotimes_{\tau} \mathrm{Sym}^{k_\tau - 2} \det{}^{1-k_\tau-\ell_\tau}(V \otimes_\tau \overline{\mathbb{F}}_p),$$

then $\overline{\rho}$ is modular of weight (k, ℓ).

Furthermore, if $k \in \Xi^{+}$, the converse holds.

Example/Evidence when $[F:\mathbb{Q}] = 2$ and $\ell = (0,0)$

Theorem (DS)

Let $2 \leq r \leq p$ and suppose that r is odd. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular.

If $\overline{\rho}_p$ has crystalline lift of weight

$$((r,1),(0,0))$$

(=HT weight $((r-1,0),(0,0))$) then $\overline{\rho}$ is (geometric) modular of weight

$$((r,1),(0,0)).$$

[We fix τ in Σ and write weights labelled by τ on the left of every pair]

Proof when $2 < r$

By p-adic Hodge theory, show that $\overline{\rho}_p$ has two crystalline lifts of weight (k, ℓ) (=HT weight $(k+\ell-1, \ell)$) and (k', ℓ')

$$(k, \ell) = ((r-1, p+1), (0, 0)) \text{ and } (k', \ell') = ((r+1, p+1), (-1, 0))$$

respectively.

By work of Gee and his collaborators on the BDJ conjecture that $\overline{\rho}$ is algebraic modular of weight (k, ℓ) (=Serre weight $V_{k,1-k-\ell}$) and (k', ℓ').

Since r and 1 are paritious, $\overline{\rho}$ is geometric modular of weight (k, ℓ) and (k', ℓ').

Let f (resp. f') be a geometric HMF of weight (k, ℓ) (resp. (k', ℓ')) such that $\overline{\rho}_f \simeq \overline{\rho}$ (resp. $\overline{\rho} \simeq \overline{\rho}_{f'}$). One observes

- $\theta_\tau(f)$ is of weight

$$((r-1, p+1), (0,0)) + ((1, p), (-1, 0)) = ((r, 2p+1), (-1, 0)),$$

- $f' H_\tau$ is an eigenform of weight

$$((r+1, p+1), (-1, 0)) + ((-1, p), (0, 0)) = ((r, 2p+1), (-1, 0)),$$

- $\theta_\tau(f) = f' H_\tau$.

Deduce from a theory of θ-operators/θ-cycles that f is divisible by H_τ. The HMF f/H_τ of weight

$$((r-1, p+1), (0, 0)) - ((-1, p), (0, 0)) = ((r, 1), (0, 0))$$

is what we are looking for.

We have seen an example of interplay between algebraic HMFs and geometric HMFs:

$$\text{Serre weight } V_{k,1-k-\ell} \leftrightsquigarrow \mathscr{A}_{(k,\ell)}$$

We will see a few more examples of this. They make intrinsic connections between mod p representations of $G(\mathbb{F}_p) = \mathrm{GL}_2(\mathscr{O}_F/p)$ and mod p geometry of the Shimura variety for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$.

Mod p representation theory

Let $\chi : (\mathcal{O}_F/p)^\times \to \overline{\mathbb{F}}_p^\times$ be a character $\prod_{\tau \in \imath \circ \Sigma} \tau^{r_\tau}$.

(Bardoe-Sin/Breuil-Paskunas)

$$I_\chi = \mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}(1 \otimes \chi)$$

has a filtration:

$$0 = I_\chi[d+1] \subset I_\chi[d] \subset \cdots \subset I_\chi[1] \subset I_\chi[0] = I_\chi$$

such that the graded piece at $0 \leq j \leq d$ is of the form

$$I_\chi[j]/I_\chi[j+1] = \bigoplus_{J \subset \Sigma, |J|=j} I_\chi^J$$

where I_χ^J is either zero or irreducible of the form

$$V_{k^J,\ell^J} = \bigotimes_\tau \det^{\ell_\tau^J} \otimes \mathrm{Sym}^{k_\tau^J - 2}\left(V \otimes_\tau \overline{\mathbb{F}}_p\right)$$

In case you are interested...

$$(\ell_\tau^J, k_\tau^J - 2) = \begin{cases} (0, r_\tau) & \text{if } \tau \notin J \text{ and } \Phi \circ \tau \notin J \\ (0, r_\tau - 1) & \text{if } \tau \in J \text{ and } \Phi \circ \tau \notin J \\ (r_\tau + 1, p - 2 - r_\tau) & \text{if } \tau \notin J \text{ and } \Phi \circ \tau \in J \\ (r_\tau, p - 1 - r_\tau) & \text{if } \tau \in J \text{ and } \Phi \circ \tau \in J \end{cases}$$

$Y_{\Gamma\cap\Gamma_0(p)}$ and $Y_{\Gamma\cap\Gamma_1(p)}$

Let $\Gamma_0(p) \subset G(\mathbb{Z}_p)$ (resp. $\Gamma_1(p) \subset G(\mathbb{Z}_p)$) denote the pre-image, by $G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p\mathbb{Z}_p)$, of $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (resp. $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$).

- (Pappas/Goren-Kassaei) $Y_{\Gamma\cap\Gamma_0(p)} = \{(A \xrightarrow{f} B)\}/\simeq$, where A and B are HBAVs of the parametrised by Y_Γ and $C = \ker f$ is an $(\mathscr{O}_F/p)$-submodule scheme of $A[p]$ such that the Weil pairing $A[p] \simeq A[p]^\vee$ induces

$$C \simeq \ker(A[p]^\vee \to C^\vee)$$

- (Pappas) $Y_{\Gamma\cap\Gamma_1(p)} = \{(A \xrightarrow{f} B, P)\}/\simeq$, where f is of the type parametrised by $Y_{\Gamma\cap\Gamma_0(p)}$ and P is an $(\mathscr{O}_F/p)$-generator of C in the sense of Drinfeld-Katz-Mazur.

Spaces of mod p HMFs of weight 2 and level $\Gamma \cap \Gamma_1(p)$

Let K be the dualising sheaf (of level obvious from the context). The (pull-back of) Kodaira-Spencer over $\overline{Y}_\Gamma$ allows use to identify K with the automorphic bundle of weight $(k, \ell) = (2, -1)$.

- $H^0(\overline{Y}_{\Gamma\cap\Gamma_0(p)}, K)$
- $H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, K)$

$$H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, K) = \bigoplus_{\chi:(\mathscr{O}_F/p\mathscr{O}_F)^\times \to \overline{\mathbb{F}}_p^\times} H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, K)_\chi$$

Mod p Jacquet-Langlands correspondence

Theorem (DKS)

There is a 'Hecke equivariant' spectral sequence

$$E_1^{j,i} = \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_\Gamma^J, \mathscr{A}_\chi^J) \Rightarrow H^{i+j}(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, K)_\chi$$

where

- G^J is $\mathrm{Res}_{F/\mathbb{Q}}$ of the units of the quaternion algebra over F ramified exactly at $\jmath \circ Q_J \subset \jmath \circ \Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$ with

$$Q_J = \{\tau \in J \,|\, \Phi \circ \tau \notin J\} \cup \{\tau \notin J \,|\, \Phi \circ \tau \in J\},$$

- $\overline{Y}_\Gamma^J$ is the $\overline{\mathbb{F}}_p$-fibre of the $\mathbb{Z}_{(p)}$-integral model of the Shimura variety of G^J with level $\Gamma \subset G^J(\mathbb{A}) \simeq G(\mathbb{A}^\infty)$,
- $\mathscr{A}_\chi^J$ is a line bundle over $\overline{Y}_\Gamma^J \leftrightsquigarrow$ the JH factor I_χ^J of I_χ

In case you are interested in $\mathscr{A}_\chi^J$...

The line bundle $\mathscr{A}_\chi^J$ over $\overline{Y}_\Gamma^J$ defined in terms of the parameters of the (non-zero) Jordan-Holder factor I_χ^J in $I_\chi = \mathrm{Ind}_{B(\mathbb{Z}_p)}^{G(\mathbb{Z}_p)} 1 \otimes \chi$, i.e., when

$$I_\chi^J = V_{(k^J,\ell^J)} = \bigotimes_\tau \det^{\ell_\tau^J} \otimes \mathrm{Sym}^{k_\tau^J - 2}\left(V \otimes_\tau \overline{\mathbb{F}}_p\right),$$

$$\mathscr{A}_\chi^J = \mathscr{A}^{Q_J} = \left[\bigotimes_{\tau \notin Q_J} \delta_\tau^{\ell_\tau^J} \otimes \omega_\tau^{k_\tau^J}\right] \otimes \left[\bigotimes_{\tau \in Q_J} \delta_\tau^{\ell_\tau^J + 1} \otimes \mathrm{Sym}^{k_\tau^J - 2} \gamma_\tau\right]$$

[γ_τ^J is the τ-part of the 'relative de Rham cohomology sheaf' over $\overline{Y}_\Gamma$ and $\delta_\tau = \bigwedge^2_{\mathscr{O}_{\overline{Y}_\Gamma^J}} \gamma_\tau$]

Mod p Jacquet-Langlands correspondence at $i+j=0$

Corollary (DKS)

There is a $(d+1)$-step 'Hecke equivariant' decreasing filtration

$$\begin{array}{ccc} H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)},K)_\chi[d+1] & \subset\cdots\subset & H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)},K)_\chi[0] \\ \| & & \| \\ 0 & & H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)},K)_\chi \end{array}$$

such that

$$H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)},K)_\chi[j]/H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)},K)_\chi[j+1] \hookrightarrow \bigoplus_{J\subset\Sigma,|J|=j} H^0(\overline{Y}^J_\Gamma,\mathscr{A}^J_\chi)$$

In case you are interested in what I mean by the spectral sequence...

- $\overline{\mathbb{F}}_p$-linear differentials

$$d_r^{i,j} : E_r^{j,i} \to E_r^{j+r,i-r+1}$$

  for $r \geq 1$, $j \geq 0$ and $i \geq -j$ with

$$E_{r+1}^{j,i} = \ker d_r^{j,i} / \mathrm{Im}\, d_r^{j+r,i-r+1}$$

- a decreasing filtration of length $d+1$ on $H^0(\overline{Y}_{\Gamma \cap \Gamma_1(p)}, K)_\chi$ and an isomorphisms

$$E_\infty^{j,i} = E_{d+1}^{j,i} \simeq H^0(\overline{Y}_{\Gamma \cap \Gamma_1(p)}, K)_\chi[j] / H^0(\overline{Y}_{\Gamma \cap \Gamma_1(p)}, K)_\chi[j+1]$$

Example when $F = \mathbb{Q}$

Suppose $F = \mathbb{Q}$.

Let $\Sigma = \{\tau\}$ and χ be a character $(\mathbb{Z}/p\mathbb{Z})^\times \xrightarrow{z \mapsto z^{r_\chi}} \overline{\mathbb{F}}_p^\times$.

$$J = \varnothing \rightsquigarrow Q_J = \varnothing \rightsquigarrow \mathscr{A}_\chi^J = \omega^{r_\chi}$$

$$J = \{\tau\} \rightsquigarrow Q_J = \varnothing \rightsquigarrow \mathscr{A}_\chi^J = \delta^{r_\chi} \otimes \omega^{p+1-r_\chi}$$

An analogous spectral sequence degenerates at E_1 and it gives rise to...

Gross's exact sequence

$$0 \to H^0(\overline{X}_\Gamma, \delta^{r_\chi} \otimes \omega^{p+1-r_\chi}(-Z_\Gamma)) \to \cdots$$

$$\cdots \to H^0(\overline{X}_{\Gamma \cap \Gamma_1(p)}, K)_\chi \to H^0(\overline{X}_\Gamma, \omega^{r_\chi+2}(-Z_\Gamma)) \to 0.$$

This is itself a geometric manifestation of the exact sequence of $\mathrm{GL}_2(\mathbb{F}_p)$-representations:

$$0 \to \det^{r_\chi} \otimes \mathrm{Sym}^{p-1-r_\chi}\mathbb{F}_p^2 \to \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} 1 \otimes \chi \to \mathrm{Sym}^{r_\chi}\mathbb{F}_p^2 \to 0.$$

Remark

We can locate the obstruction for the injection to be an isomorphism, with an explicit example. If I venture my guess, the obstruction is of ‘Eisenstein’.

Finally, one can also prove the following vanishing result for relative cohomology of the dualising sheaf:

Theorem (DKS)

Let $\pi : Y_{\Gamma \cap \Gamma_1(p)} \to Y_\Gamma$ be the natural projection on the integral $\mathbb{Z}_p$-models for G. Then

$$R^r \pi_* K = 0$$

for $r > 0$.

This is used in our construction of mod p automorphic Galois representations above.