p-ADIC GROSS–ZAGIER FORMULA AT CRITICAL SLOPE AND A CONJECTURE OF PERRIN-RIOU

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Abstract. Let p be an odd prime. Given an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D_K})$ where p splits with $D_K > 3$, and a p-ordinary newform $f \in S_k(\Gamma_0(N))$ such that N verifies the Heegner hypothesis relative to K, we prove a p-adic Gross–Zagier formula for the critical slope p-stabilization of f (assuming that it is non-θ-critical). In the particular case when $f = f_A$ is the newform of weight 2 associated to an elliptic curve A that has good ordinary reduction at p, this allows us to verify a conjecture of Perrin-Riou. The p-adic Gross–Zagier formula we prove has applications also towards the Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one.

Contents

Key words and phrases. Heegner cycles, Families of p-adic modular forms, Birch and Swinnerton-Dyer conjecture.

1. Introduction

Fix forever an odd prime p as well as embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let N be an integer coprime to p. We let v_p denote the valuation on $\overline{\mathbb{Q}}_p$, normalized so that $v_p(p) = 1$.

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ be a newform of even weight $k \geq 2$ and level $N \geq 3$. Let $K_f := \iota_\infty^{-1}(\mathbb{Q}(\cdots, a_n, \cdots))$ denote the Hecke field of f and $\mathfrak{P}$ the prime of K_f induced by the embedding ι_p. Let E denote an extension of $\mathbb{Q}_p$ that contains $\iota_p(K_f)$. We shall assume that $v_p(\iota_p(a_p)) = 0$, namely that f is $\mathfrak{P}$-ordinary. Let $\alpha, \beta \in \overline{\mathbb{Q}}$ denote the roots of the Hecke polynomial $X^2 - \iota_\infty^{-1}(a_p)X + p^{k-1}$ of f at p. We assume that E is large enough to contain both $\iota_p(\alpha)$ and $\iota_p(\beta)$. Since we assume that f is $\mathfrak{P}$-ordinary, precisely one of $\iota_p(\alpha)$ and $\iota_p(\beta)$ (say, without loss of generality, $\iota_p(\alpha)$) is a p-adic unit. Then $v_p(\iota_p(\beta)) = k - 1$. To ease our notation, we will omit ι_p and ι_∞ from our notation unless there is a danger of confusion.

The p-stabilization $f^\alpha \in S_k(\Gamma_0(Np))$ of f is called the ordinary stabilization and f^β is called the critical-slope p-stabilization. We shall assume throughout that f^β is not θ-critical (in the sense of Definition 2.12 in [Bel12]).

Our main goal in the current article is to prove a p-adic Gross–Zagier formula for the critical-slope p-stabilization f^β. This is Theorem 1.1.1. In the particular case when f has weight 2 and it is associated to an elliptic curve $A_{/\mathbb{Q}}$, this result allows us to prove a conjecture of Perrin-Riou. This is recorded below as Theorem 1.1.5; it can be also translated into the statement of Theorem 1.1.4, which is an explicit construction of a point of infinite order in $A(\mathbb{Q})$ in terms of the two p-adic L-functions associated to f_A (under the assumption that A has analytic rank one, of course). As a by product of Theorem 1.1.1, we may also deduce that at least one of the two p-adic height pairings associated to A is non-degenerate. This fact yields the proof of the p-part of the Birch and Swinnerton-Dyer formula[1] for A; this is Theorem 1.1.8 below.

Before we discuss these results in detail, we will introduce more notation. Let S denote the set consisting of all rational primes dividing Np together with the archimedean place. We let W_f denote Deligne's (cohomological) p-adic representation associated to f (so that the Hodge–Tate weights of W_f are $(1-k,0)$, with the convention that the Hodge–Tate weight of the cyclotomic character is $+1$). Set $V_f = W_f(k/2)$; we call V_f the central critical twist of W_f. Both W_f and V_f are unramified outside S and they are crystalline at p.

Let $\mathbb{D}_{\mathrm{cris}}(V_f)$ denote the crystalline Dieudonné module and $\mathbb{D}^{\dagger}_{\mathrm{rig}}(V_f)$ Fontaine's (étale) $(\varphi,\Gamma_{\mathrm{cyc}})$-module associated to $V_f|_{G_{\mathbb{Q}_p}}$. We let D_α, D_β denote the eigenspaces of $\mathbb{D}_{\mathrm{cris}}(V_f)$ for the action of the crystalline Frobenius φ; so that $\varphi|_{D_\alpha} = p^{-k/2}\alpha$ and $\varphi|_{D_\beta} = p^{-k/2}\beta$.

Let $K=\mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field and let $H^1_{\mathrm{f}}(K,V_f)$ denote the Bloch-Kato Selmer group associated to V_f. For each $\lambda\in\{\alpha,\beta\}$ the submodule $D_\lambda\subset\mathbb{D}_{\mathrm{cris}}(V_f)$ defines a canonical splitting of the Hodge filtration on $\mathbb{D}_{\mathrm{cris}}(V_f)$, namely that we have

$$\mathbb{D}_{\mathrm{cris}}(V_f) = D_\lambda \oplus \mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V_f).$$

as E-vector spaces. Note that our assumption that f^β is non-θ-critical is necessary to ensure this splitting when $\lambda=\beta$ (see [Bel12, Proposition 2.11(iv)]). We let

$$h^{\mathrm{Nek}}_{\lambda,K} : H^1_{\mathrm{f}}(K,V_f)\times H^1_{\mathrm{f}}(K,V_f)\longrightarrow E$$

denote the p-adic height pairing that Nekovář in [Nek93] has associated to this splitting.

Suppose that the prime p splits in K and write $(p)=\mathfrak{p}\mathfrak{p}^c$. Assume also that K verifies the Heegner hypothesis relative to N. Let ϵ_K denote the quadratic Dirichlet character associated to $K/\mathbb{Q}$. The Heegner hypothesis ensures that $\mathrm{ord}_{s=\frac{k}{2}}L(f_{/K},s)$ is odd and there exists a Heegner cycle $z_f\in H^1_{\mathrm{f}}(K,V_f)$.

1.1. **Results.** Let $L^{\mathrm{Kob}}_{p,\beta}(f_{/K},s)$ be the p-adic L-function given as in (4.3.1). It is the critical slope counterpart of Nekovář's p-adic L-function associated to the p-ordinary stabilization f^α. It follows from its interpolation property that $L^{\mathrm{Kob}}_{p,\beta}(f_{/K},1)=0$. As its predecessors, our p-adic Gross–Zagier formula expresses the first derivative of $L^{\mathrm{Kob}}_{p,\beta}(f_{/K},s)$ in terms of the p-adic height of the Heegner cycle z_f:

Theorem 1.1.1. *Let $f\in S_k(\Gamma_0(N))$ be a newform with $N\geq 3$. Suppose f is p-ordinary with respect to the embedding ι_p and let f^β denote its critical-slope p-stabilization (of slope $v_p(\beta)=k-1$). Assume also that f^β is not θ-critical. Let $K=\mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field where the prime p splits and that satisfies the Heegner hypothesis relative to N. Then,*

$$\frac{d}{ds}L^{\mathrm{Kob}}_{p,\beta}(f_{/K},s)\big|_{s=\frac{k}{2}} = \left(1-\frac{p^{\frac{k}{2}-1}}{\beta}\right)^4\cdot\frac{h^{\mathrm{Nek}}_{\beta,K}(z_f,z_f)}{(4|D_K|)^{\frac{k}{2}-1}}.$$

[1]Under the additional hypothesis that A be semistable, this has been proved in [BBV16, JSW17, Zha14] using different techniques. We do not need to assume that A is semistable.

This theorem is proved by appealing to the existence of p-adic families of finite slope modular forms, which allows us, using the existence of a suitable two-variable p-adic L-function[2] and Theorem 3.5.4 to reduce to the case of non-critical slope. Such a non-critical slope result is precisely Theorem 5.1.1 (p-adic Gross-Zagier formula for non-ordinary eigenforms of arbitrary weight; which is work in progress by Kobayashi [Kob19]). More precisely, Kobayashi's method only establishes a p-adic Gross–Zagier formula in the non-ordinary case for one of the two p-stabilization of a given form (the one of smaller slope). However, this result is sufficient for our method and moreover, our method not only yields the Gross–Zagier formula in the case of critical slope, but also allows us to handle the case of the other non-ordinary p-stabilization. See Theorem 1.1.11 for an even more general statement.

1.1.1. *Abelian varieties of* GL_2*-type.* We assume until the end of this introduction that f has weight 2. Let $A_f/\mathbb{Q}$ denote the abelian variety of GL_2-type that the Eichler-Shimura congruences associate to f. This means that there exists an order $\mathcal{O}_f \subset K_f$ and an embedding $\mathcal{O}_f \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A_f)$. We shall assume that $\mathrm{ord}_{s=1}L(f_{/\mathbb{Q}},1) = 1$ and we choose K (relying on [BFH90]) in a way to ensure that $\mathrm{ord}_{s=1}L(f_{/K},1) = 1$ as well. In this scenario, the element $z_f \in H^1(\mathbb{Q},V_f)$ is obtained as the Kummer image of the f-isotypical component P_f of a *Heegner point*[3] $P \in J_0(N)(K)$. Here, $J_0(N)$ is the Jacobian variety of the modular curve $X_0(N)$ and we endow it with the canonical principal polarization induced by the intersection form on $H^1(X_0(N),\mathbb{Z})$. This equips A_f with a canonical polarization as well.

We let $\langle\, ,\,\rangle_\infty^{J_0(N)}$ denote the Néron-Tate height pairing on the abelian variety $J_0(N)$. Nekovář's constructions in [Nek93] gives rise to a pair of E-equivariant p-adic height pairings

$$h^{\mathrm{Nek}}_{\lambda,\mathbb{Q}} : (A_f(\mathbb{Q})\otimes_{\mathcal{O}_f} E) \times (A_f(\mathbb{Q})\otimes_{\mathcal{O}_f} E) \longrightarrow E$$

for each $\lambda = \alpha,\beta$. We set

$$c(f) := -\frac{L'(f_{/\mathbb{Q}},1)}{\langle P_f,P_f\rangle_\infty^{J_0(N)} 2\pi i\,\Omega_f^+} \in K_f^\times$$

where Ω_f^+ is a choice of Shimura's period. We note that K_f-rationality of $c(f)$ is proved in [GZ86].

Corollary 1.1.2. *In addition to the hypotheses of Theorem 1.1.1, suppose that* $k = 2$ *and* $\mathrm{ord}_{s=1}L(f_{/\mathbb{Q}},1) = 1$. *Then for* A_f, $P_f \in A_f(\mathbb{Q})$ *and* $c(f) \in K_f^\times$ *as in the previous paragraph we have*

$$L'_{p,\beta}(f_{/\mathbb{Q}},1) = (1-1/\beta)^2\, c(f)\, h^{\mathrm{Nek}}_{\beta,\mathbb{Q}}(P_f,P_f).$$

Remark 1.1.3. The version of Theorem 1.1.1 above for the p-ordinary stabilization f^α is due to Perrin-Riou (when $k = 2$) and Nekovář (when k is general). The version of Corollary 1.1.2 concerning the p-adic L-function $L_{p,\alpha}(f_{/\mathbb{Q}},s)$ follows from Perrin-Riou's p-adic Gross–Zagier theorem.

[2]The construction of this p-adic L-function follows from the work of Loeffler [Loe17] and Loeffler-Zerbes [LZ16].

[3]More precisely, $P \in J_0(N)(K)$ is given as the trace of a Heegner point $y \in J_0(N)(H_K)$ which is defined over the Hilbert class field H_K of K. Our restriction on the sign of the functional equation (for the Hecke L-function of f) shows that that $P_f \in J_0(N)(\mathbb{Q}) \otimes K_f$.

1.1.2. *Elliptic curves.* In this subsection, we will specialize to the case when $K_f = \mathbb{Q}$, so that $A = A_f$ is an elliptic curve defined over $\mathbb{Q}$ of conductor N and analytic rank one, with good ordinary reduction at p and without CM. We note that it follows from [Eme04, Theorem 1.3] that f^β is not θ-critical.

We still assume that $\mathrm{ord}_{s=1} L(f_{/\mathbb{Q}}, 1) = 1$ and we choose K as in Section 1.1.1. We assume that the mod p representation

$$\overline{\rho}_A : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{F}_p}(A[p]) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{F}_p)$$

is absolutely irreducible. We fix a Weierstrass minimal model $\mathcal{A}_{/\mathbb{Z}}$ of A and let $\omega_{\mathcal{A}}$ denote the Néron differential normalized as in [PR95, §3.4] and is such that its associated real period Ω_A^+ is positive. Set $V = T_p(A) \otimes \mathbb{Q}_p$ and we let $\omega_{\mathrm{cris}} \in \mathbb{D}_{\mathrm{cris}}(V)$ denote the element that corresponds to $\omega_{\mathcal{A}}$ under the comparison isomorphism. Extending scalars (to a sufficiently large extension E of $\mathbb{Q}_p$) if need be, we shall denote by $D_\alpha, D_\beta \subset \mathbb{D}_{\mathrm{cris}}(V)$ the corresponding eigenspaces as before. Set $\omega_{\mathrm{cris}} = \omega_\alpha + \omega_\beta$ with $\omega_\alpha \in D_\alpha$ and $\omega_\beta \in D_\beta$. We let

$$[-,-] : \mathbb{D}_{\mathrm{cris}}(V) \times \mathbb{D}_{\mathrm{cris}}(V) \longrightarrow E$$

denote the canonical pairing (induced from the Weil pairing) and we set $\delta_A := [\omega_\beta, \omega_\alpha]/c(f)$. We let $\omega_{\mathcal{A}}^* \in \mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V)$ denote the unique element such that $[\omega_{\mathcal{A}}, \omega_{\mathcal{A}}^*] = 1$. We remark that $\mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V)$ may be identified with the tangent space of $A(\mathbb{Q}_p)$ and the Bloch-Kato exponential map

$$\exp_V : \mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V) \longrightarrow H^1_{\mathrm{f}}(\mathbb{Q}_p, V) = A(\mathbb{Q}_p) \otimes \mathbb{Q}_p$$

with the exponential map for the p-adic Lie group $A(\mathbb{Q}_p)$.

Theorem 1.1.4. *Suppose $A = A_f$ is in the previous paragraph (so $k = 2$, $K_f = \mathbb{Q}$ and $\overline{\rho}_f$ is absolutely irreducible). In addition to all the hypotheses of Theorem 1.1.1, assume that* $\mathrm{ord}_{s=1} L(A_{/\mathbb{Q}}, 1) = 1$. *Then*

$$\exp_V\left(\omega_{\mathcal{A}}^* \cdot \sqrt{\delta_A\left((1-1/\alpha)^{-2} \cdot L'_{p,\alpha}(f_{/\mathbb{Q}}, 1) - (1-1/\beta)^{-2} \cdot L'_{p,\beta}(f_{/\mathbb{Q}}, 1)\right)}\right)$$

is a $\mathbb{Q}$-rational point on the elliptic curve A of infinite order.

The theorem above asserts the validity of a conjecture of Perrin-Riou. We also note that this theorem allows for the explicit computation of rational points on elliptic curves. Indeed one can compute the expression appearing in Theorem 1.1.4 to very high p-adic accuracy by using the methods of [PS11] where algorithms are given to compute the derivatives of both ordinary and critical slope p-adic L-functions. Such computations should be compared to the analogous computations in [KP07] in the non-ordinary case.

Theorem 1.1.4 may be deduced from the next result we present (in a manner identical to the argument in [Büy17, §2.3]), which compares the Bloch-Kato logarithms of two distinguished elements of the Bloch-Kato Selmer group $H^1(\mathbb{Q}, V)$: the Beilinson-Kato element BK_1 and the Heegner point P_f given as above, for an appropriate choice of the imaginary quadratic field K. Notice that under our running hypotheses

$$H^1_{\mathrm{f}}(\mathbb{Q}, V) = A(\mathbb{Q}) \otimes \mathbb{Q}_p$$

and it is a one dimensional $\mathbb{Q}_p$-vector space. Note that $P_f \in A(\mathbb{Q})$ is a rational point on A and as such, it is a genuinely algebraic object, whereas $\mathrm{BK}_1 \in A(\mathbb{Q}) \otimes \mathbb{Q}_p$ is a transcendental object that relates to both p-adic L-functions. The proof of Theorem 1.1.4 boils down to setting up an explicit comparison between BK_1 and P_f. This is precisely the content of Theorem 1.1.5. It was conjectured by Perrin-Riou and was proved independently by Bertolini–Darmon–Venerucci in their preprint [BDV19] (their approach is different from ours).

Theorem 1.1.5. *Suppose $A_{/\mathbb{Q}}$ is an elliptic curve as in Theorem 1.1.4 and let $P \in A(\mathbb{Q})$ be a generator of the free part of its Mordell-Weil group. We have*

$$\log_A(\mathrm{res}_p(\mathrm{BK}_1)) = -(1-1/\alpha)(1-1/\beta)\cdot c(f)\cdot \log_A(\mathrm{res}_p(P))^2\,,$$

where $\log_A$ stands for the coordinate of the Bloch-Kato logarithm associated to A with respect to the basis (of the tangent space) dual to that given by the Néron differential $\omega_{\mathcal{A}}$.

One key result that we rely on establishing Theorem 1.1.5 is the following consequence of our p-adic Gross–Zagier formula. We record it in Section 1.1.3 as we believe that it is of independent interest; while we re-iterate that a proof of Theorem 1.1.4 is not written down explicitly in this article as it follows verbatim as in [Büy17].

1.1.3. *p-adic heights on Abelian varieties of* GL_2*-type and the conjecture of Birch and Swinnerton-Dyer.* Throughout this section, we still assume that $f \in S_2(\Gamma_0(N))$ has weight two; but we no longer assume that $K_f = \mathbb{Q}$. We retain our hypothesis that $\mathrm{ord}_{s=1} L(f_{/\mathbb{Q}}, 1) = 1$ and we choose K as in Section 1.1.1. In this situation, it follows from the work of Gross–Zagier and Kolyvagin–Logachev that the Tate-Shafarevich group $\mathrm{III}(A_f/\mathbb{Q})$ is finite and the Heegner point

$$P := \sum_{\sigma: K_f \hookrightarrow \overline{\mathbb{Q}}} P_{f^\sigma} \in A_f(\mathbb{Q})$$

generates $A_f(\mathbb{Q}) \otimes \mathbb{Q}$ as a K_f-vector space.

Theorem 1.1.6. *Suppose $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ is a newform with $N \geq 3$ and such that*

- $v_p(\iota_p(a_p)) = 0$,
- *neither of the p-stabilizations of f is θ-critical,*
- *the residual representation $\overline{\rho}_f$ (associated to the $\mathfrak{P}$-adic representation attached to f) is absolutely irreducible,*
- $\mathrm{ord}_{s=1} L(f_{/\mathbb{Q}}, 1) = 1$.

Then either $h^{\mathrm{Nek}}_{\alpha,\mathbb{Q}}$ or $h^{\mathrm{Nek}}_{\beta,\mathbb{Q}}$ is non-degenerate.

Remark 1.1.7. When $K_f = \mathbb{Q}$ and p is a prime of good supersingular reduction for the elliptic curve $A = A_f$, a stronger form of Theorem 1.1.6 was proved by Kobayashi in [Kob13]. Fortunately, this weaker version is good enough for applications towards the Birch and Swinnerton-Dyer conjecture we discuss below.

The final result we shall record in this introduction (Theorem 1.1.8 below) is a consequence of Theorem 1.1.1 and Theorem 1.1.6 towards the Birch and

Swinnerton-Dyer conjecture for the abelian variety A_f. Under the additional hypothesis that $K_f = \mathbb{Q}$ and A be semistable, this has been proved in [BBV16, JSW17, Zha14][4] using different techniques. Our results here allow us to adapt the proof of [Kob13, Cor. 1.3]) to the current setting to obtain a much simpler proof of (the p-part of) the Birch and Swinnerton-Dyer formula and eliminate the semistability hypothesis in [JSW17].

Before we state our result, we define the $\mathcal{O}_f$-equivariant L-function $L(A_f/\mathbb{Q}, s)$ (with values in $K_f \otimes \mathbb{C}$) by setting

$$L(A_f/\mathbb{Q}, s) := (L(f^\sigma/\mathbb{Q}, s))_{\sigma\in\Sigma}\,,$$

where $\Sigma = \{\sigma : K_f \hookrightarrow \overline{\mathbb{Q}}\}$. For any $\{x_i\} \subset A_f(\mathbb{Q})$ (resp. $\{y_j\} \subset A_f^\vee(\mathbb{Q})$) that induces a basis of $A_f(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. of $A_f^\vee(\mathbb{Q}) \otimes \mathbb{Q}$), the Néron-Tate regulator $R_\infty(A_f/\mathbb{Q})$ on $A_f(\mathbb{Q})$ is given as

$$R_\infty(A_f/\mathbb{Q}) := \frac{\det(\langle x_i, y_j\rangle_\infty)}{[A_f(\mathbb{Q}) : \sum \mathbb{Z}x_i]\,[A_f^\vee(\mathbb{Q}) : \sum \mathbb{Z}y_j]}.$$

We let $\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})$ denote the σ-component of this regulator, given as in (14), so that we have

$$\mathrm{Reg}_\infty(A_f/\mathbb{Q}) = \prod_{\sigma\in\Sigma} \mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})$$

(see Remark 6.2.2 where we discuss this factorization). We may then write

$$L^*(A_f/\mathbb{Q}, 1) := \left(-\frac{L'(f^\sigma/\mathbb{Q}, 1)}{\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})\cdot 2\pi i\,\Omega^+_{f^\sigma}}\right)_{\sigma\in\Sigma} \in K_f \otimes \overline{\mathbb{Q}}$$

to denote the algebraic part of the leading coefficient of the equivariant L-function $L(A_f/\mathbb{Q}, s)$ at $s = 1$.

Theorem 1.1.8. *Suppose $f \in S_2(\Gamma_0(N))$ is a newform as in Theorem 1.1.6. If the Iwasawa main conjecture holds true for each $f^\sigma_{/\mathbb{Q}}$, we have*

$$L^*(A_f/\mathbb{Q}, 1) \in \frac{|\text{Ш}(A_f/\mathbb{Q})|\cdot \mathrm{Tam}(A_f/\mathbb{Q})}{|A_f(\mathbb{Q})_{\mathrm{tor}}|\cdot|A_f^\vee(\mathbb{Q})_{\mathrm{tor}}|}\,(\mathcal{O}_f\otimes\mathbb{Z}_p)^\times\,.$$

Here:

- $\mathrm{Tam}(A_f/\mathbb{Q}) := \prod_{\ell|N} c_\ell(A_f/\mathbb{Q})$ *and* $c_\ell(A_f/\mathbb{Q})$ *is the Tamagawa factor at* ℓ.
- $A_f(\mathbb{Q})_{\mathrm{tor}}$ *(resp.* $A_f^\vee(\mathbb{Q})_{\mathrm{tor}}$*) is the torsion subgroup of the Mordell-Weil group of* A_f *(resp. of the dual abelian variety* $A_f^\vee$*).*

Corollary 1.1.9. *Suppose $A_{/\mathbb{Q}}$ is a non-CM elliptic curve with analytic rank one and that*

(MC1) *A has good ordinary reduction at p,*
(MC2) *$\overline{\rho}_A$ is absolutely irreducible,*
(MC3) *one of the following two conditions hold:*

[4] Besides the assumption that A be semistable, [Zha14, Theorem 7.3] has additional assumption that p is coprime to Tamagawa factors and [BBV16, Theorem A] requires p be non-anomalous for A. In Section 5.6 of [BBV16], the authors explain a strategy to weaken the semistability hypothesis.

(MC3.1) *There exists a prime* $q||N$ *such that* $p \nmid \mathrm{ord}_q(\Delta_q)$ *for a minimal discriminant* Δ_q *of* A *at* q.

(MC3.2) *We have* $\rho_A(G_{\mathbb{Q}}) \supset \mathrm{SL}_2(\mathbb{Z}_p)$ *and there exists a real quadratic field* F *verifying the conditions of* [Wan15, Theorem 4].

Then the p*-part of the Birch and Swinnerton-Dyer formula for* A *holds true.*

Remark 1.1.10. The Iwasawa main conjecture for $f^{\sigma}_{/\mathbb{Q}}$ relates the characteristic ideal of a Selmer group of the p-adic Galois representation $T^{\sigma} := \varprojlim A_f(\overline{\mathbb{Q}})[\mathfrak{P}_{\sigma}^n]$, where $\mathfrak{P}_{\sigma}$ is the prime of K_f that is induced by the embedding $\iota_p \circ \sigma : K_f \hookrightarrow \overline{\mathbb{Q}}_p$, to one of the p-adic L-functions $L_{p,\lambda^{\sigma}}(f^{\sigma}_{/\mathbb{Q}}, s)$ (where $\lambda^{\sigma} := \iota_p \circ \sigma(\lambda)$ for $\lambda = \alpha$ or β and where we have extended σ to an embedding $K_f(\alpha) \hookrightarrow \overline{\mathbb{Q}}$ in an arbitrary manner). Whether or not T^{σ} is an ordinary Galois representation or not depends on whether or not $\iota_p \circ \sigma \circ \iota_{\infty}^{-1}(a_p)$ is a p-adic unit and therefore, the proof of the p-part of Birch and Swinnerton-Dyer formula for a general GL_2-type abelian variety requires the Iwasawa main conjecture both for primes of good ordinary reduction and good supersingular reduction. There has been great progress in this direction; c.f. the works of Skinner-Urban and Wan.

When $K_f = \mathbb{Q}$ and p is a prime of good ordinary reduction for $A = A_f$, one only needs the main conjectures for a good ordinary prime. This has been proved in [SU14] and [Ski16, Theorem 2.5.2] (under the hypotheses (MC1), (MC2) and (MC3.1)) and in [Wan15] (under (MC1) and (MC3.2)).

We close this introduction with a brief overview of our strategy to prove Theorem 1.1.1. We remark that the original approach of Perrin-Riou and Kobayashi (which is an adaptation of the original argument of Gross and Zagier) cannot be applied in our case of interest as there is no Rankin-Selberg construction of the critical-slope p-adic L-functions $L_p(f^{\beta}_{/\mathbb{Q}}, s)$ and $L_p(f^{\beta}_{/\mathbb{Q}} \otimes \epsilon_K, s)$. The main idea is to prove a version of the asserted identity in p-adic families. That is to say, we shall choose a Coleman family $\underline{\mathbf{f}}$ through the p-stabilized eigenform f^{β} (over an affinoid domain $\mathscr{A}$) and we shall consider the following objects that come associated to $\underline{\mathbf{f}}$:

- A two-variable p-adic L-function $L_p(\underline{\mathbf{f}}_{/K}, s)$. The construction is essentially due to Loeffler (and it compares to that due to Bellaïche); we recall its defining properties in Section 4 below. One subtle point is that this p-adic L-function does not[5] interpolate $L_{p,\beta}^{\mathrm{Kob}}(f_{/K}, s)$, but rather an explicit multiple of it. This extra (non-interpolatable) p-adic multiplier is essentially the p-adic interpolation factor for the adjoint p-adic L-function attached to f^{β}. Crucially, the same factor also appears in the height side.
- An $\mathscr{A}$-adic height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$ that interpolates Nekovář's p-adic height pairings for the members of the Coleman family, in the sense that the diagram (4) below (located just before the start of Section 3.5) commutes. It is important to compare the "correction factor"[6] that appears on the right

[5]In fact, it could not: See Remark 4.3.2 below where we explain that $L_{p,\beta}^{\mathrm{Kob}}(f_{/K}, s)$ does not vary continuously as f^{β} varies in families.

[6]This factor appears as the ratio of the two Poincaré duality pairings on the the f-direct summand summands of two modular curves of respective levels $\Gamma_0(N)$ and $\Gamma_0(N) \cap \Gamma_1(p)$. See

most vertical arrow in the lower right square to the non-interpolatable p-adic multiplier mentioned in the previous paragraph. The construction of the $\mathscr{A}$-adic height pairing is due to Benois and it is recalled in Section 3.3 below.

- A "universal" Heegner point $\mathscr{Z}_{\underline{\mathbf{f}}}$ that interpolates the Heegner cycles associated to the central critical twists of the members of the family $\underline{\mathbf{f}}$. The construction of this class is one of the main ingredients here and it is carried out in [JLZ19, BL19].

Relying on the density of non-critical-slope crystalline points in the family $\underline{\mathbf{f}}$ and a p-adic Gross–Zagier formula for these members (recorded in Theorem 5.1.1, which is Kobayashi's work in progress), one may easily deduce an $\mathscr{A}$-adic Gross–Zagier formula[7] for $L_p(\underline{\mathbf{f}}_{/K}, s)$, expressing its derivative with respect to the cyclotomic variable as the $\mathscr{A}$-adic height of the universal Heegner cycle (see Theorem 5.2.2 below). The proof of Theorem 1.1.1 follows, on specializing this statement to weight k.

Let $g = \sum_{n=1} a_n(g)q^n \in S_{2r}(\Gamma_0(N))$ be a normalized eigenform. We let $a, b \in \overline{\mathbb{Q}}$ denote the roots of its Hecke polynomial $X^2 - a_p(g)X + p^{2r-1}$ at p. Suppose that $v_p(\iota_p(a_p(g))) > 0$ and assume that

$$0 < v_p(\iota_p(b)) \leq v_p(\iota_p(a))\,.$$

Let $g^b \in S_{2r}(\Gamma_0(Np))$ denote the p-stabilization corresponding to the Hecke root b. Kobayashi's forthcoming result (Theorem 5.1.1 below) proves a p-adic Gross–Zagier formula for the p-stabilization g^b alone. This is sufficient for our purposes; moreover, the method we present here (without any modification whatsoever) allows one to deduce the following p-adic Gross–Zagier formula at every non-θ-critical point x on the eigencurve of tame level N, that admits a neighborhood with a dense set of crystalline classical points (e.g., any crystalline non-θ-critical classical point x verifies this property).

Theorem 1.1.11. *Suppose x is any non-θ-critical point of weight w on the eigencurve of tame level $N \geq 3$, that admits a neighborhood with a dense set of crystalline classical points. Set $L_p(x,s) := L_p^{\mathrm{RS}}(\mathbf{F},\kappa,s)|_x$, where $\mathbf{F}$ is any Coleman family over a sufficiently small neighborhood of x and finally $L_p^{\mathrm{RS}}(\mathbf{F},\kappa,s)$ is as in Definition 4.1.4. Then,*

$$\frac{d}{ds}L_p(x,s)|_{s=\frac{w}{2}} = H_{x,K}(\mathscr{Z}_x, \mathscr{Z}_x)\,.$$

Here, $H_{x,K}$ is the specialization of the height pairing $H_{\mathbf{F},K}$ (given as in Definition 5.2.1) to x and likewise, $\mathscr{Z}_x$ is the specialization of the universal Heegner cycle $\mathscr{Z}_{\mathbf{F}}$ to x. In particular, if $f \in S_k(\Gamma_0(N))$ is a classical newform and λ is a φ-eigenvalue on $\mathbb{D}_{\mathrm{cris}}(V_f)$ such that f^λ is non-θ-critical, then

$$\frac{d}{ds}L^{\mathrm{Kob}}_{p,\lambda}(f_{/K},s)\big|_{s=\frac{k}{2}} = \left(1-\frac{p^{\frac{k}{2}-1}}{\lambda}\right)^4 \cdot \frac{h^{\mathrm{Nek}}_{\lambda,K}(z_f,z_f)}{(4|D_K|)^{\frac{k}{2}-1}}.$$

Proposition 3.4.5 where we make this discussion precise. We are grateful to D. Loeffler for explaining this to us.

[7]See also [Dis19] where a similar formula for slope-zero families was proved independently.

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2. Notation and Set up

For any field L, we let $\overline{L}$ denote a fixed separable closure and let $G_L := \mathrm{Gal}(\overline{L}/L)$ denote its Galois group.

For each prime λ of a number field F, we fix a decomposition group at λ and identify it with $G_\lambda := G_{F_\lambda}$. We denote by $I_\lambda \subset G_\lambda$ the inertia subgroup. In the main body of our article, we will only work with the case when $F = \mathbb{Q}$ or $F = K$ (the imaginary quadratic field we have fixed above). For any finite set of places S of F, we denote by F_S the maximal extension of F unramified outside S and set $G_{F,S} := \mathrm{Gal}(F_S/F)$.

We set $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$, the p-adic completion of $\overline{\mathbb{Q}}_p$. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. When the prime p is assumed to split in the imaginary quadratic field K, we let $\mathfrak{p}$ denote the prime of K corresponding to the embedding ι_p.

We denote by $v_p : \mathbb{C}_p \to \mathbb{R} \cup \{+\infty\}$ the p-adic valuation on $\mathbb{C}_p$ which is normalized by the requirement that $v_p(p) = 1$. Set $|x|_p = p^{-v_p(x)}$.

We fix a system $\varepsilon = (\zeta_{p^n})_{n\geqslant 1}$ of primitive p^nth roots of the unity in $\overline{\mathbb{Q}}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all n. We set $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ and denote by

$$\chi_{\mathrm{cyc}} : \Gamma_{\mathrm{cyc}} \xrightarrow{\sim} \mathbb{Z}_p^\times$$

the cyclotomic character. The group Γ_{cyc} factors canonically as $\Gamma_{\mathrm{cyc}} = \Delta \times \Gamma$ where $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}(\zeta_p))$. We let ω denote the Teichmüller character (that factors through Δ) and set $\langle\chi_{\mathrm{cyc}}\rangle := \omega^{-1}\chi_{\mathrm{cyc}}$. We let $\Lambda := \mathbb{Z}_p[[\Gamma]]$. We write Λ^ι to denote the free Λ-module of rank one, on which $G_{\mathbb{Q}}$ acts via

$$\begin{gathered} G_{\mathbb{Q}} \twoheadrightarrow \Gamma \xrightarrow{\iota} \Gamma \hookrightarrow \Lambda^\times \\ \iota : \gamma \longmapsto \gamma^{-1} \end{gathered}$$

By slight abuse, we denote all the objects ($\Gamma_{\mathrm{cyc}}, \chi_{\mathrm{cyc}}, \Delta, \Gamma, \omega, \Lambda$ and ι) introduced in the previous paragraph but defined over the base field $\mathbb{Q}_p$ (in place of $\mathbb{Q}$) with the same set of symbols.

For any a topological group G and a module M that is equipped with a continuous G-action, we shall write $C^\bullet(G,M)$ for the complex of continuous cochains of G with coefficients in M.

Let S be a finite set of places of $\mathbb{Q}$ that contains p and prime at infinity. If V is a p-adic representation of $G_{\mathbb{Q},S}$ with coefficients in an affinoid algebra A, we shall denote by $\mathbb{D}^\dagger_{\mathrm{rig},A}(V)$ the $(\varphi,\Gamma_{\mathrm{cyc}})$-module associated to the restriction of V to the decomposition group at p.

Let Σ denote the set of rational primes that divides Np, together with the archimedean place. We denote the set of places of K above those in Σ also by Σ.

For our fixed imaginary quadratic field K, we let $\mathcal{O}$ denote the maximal order of K. For any positive integer c, let $\mathcal{O}_c := \mathbb{Z} + c\mathcal{O}$ denote the order of conductor c in K and let H_c denote the ring class field of K of $\mathcal{O}_c$. Write $H_c^{(Np)}$ for the maximal extension of H_c outside Np and $\mathfrak{G}_c := \mathrm{Gal}(H_c^{(Np)}/H_c)$. Fix a positive integer c coprime to N. We also set $L_{cp^s} := H_{cp^s}(\mu_{p^s})$.

For any eigenform g, we shall write g^K in place of $g \otimes \epsilon_K$ for its twist by the quadratic character associated to $K/\mathbb{Q}$.

For each non-negative real number h, we let $\mathcal{D}_h$ denote the $\mathbb{Q}_p$-vector space of h-tempered distributions on $\mathbb{Z}_p$ and set $\mathcal{D}_\infty := \cup_h \mathcal{D}_h$. We also let $\mathcal{D}$ denote the Λ-algebra of $\mathbb{Q}_p$-valued locally analytic distributions on $\mathbb{Z}_p$. The natural map $\mathcal{D}_h \to \mathcal{D}$ is an injection (for every h) since locally analytic functions are dense in the space of continuous functions.

We let $\mathscr{R}_+$ denote the $\mathbb{Q}_p$-algebra of analytic functions on the open unit ball. In explicit terms,

$$\mathscr{R}_+ := \left\{ \sum_{n=0}^{\infty} c_n X^n : \lim_{n\to\infty} |c_n|_p s^n = 0 \text{ for every } s \in [0,1) \right\}.$$

According to [PR94, Proposition 1.2.7], the algebra $\mathcal{D}$ is naturally isomorphic via the Amice transform to $\mathscr{R}_+$.

On fixing a topological generator γ of Γ (which in turn fixes isomorphisms $\Gamma \cong \mathbb{Z}_p$ and $\Lambda \cong \mathbb{Z}_p[[X]]$), we may define the $\mathbb{Q}_p$-algebras

$$\mathcal{D}_h(\Gamma) \subset \mathcal{D}_\infty(\Gamma) \subset \mathcal{D}(\Gamma)$$

of distributions on Γ. We also set

$$\mathscr{H} := \{ f(\gamma-1) : f \in \mathscr{R}_+ \} \subset \mathbb{Q}_p[[\Gamma]]$$

(so that $\mathscr{H} \cong \mathscr{R}_+$ via $\gamma \mapsto 1+X$). For $H = \sum_{n=0}^\infty c_n(H)(\gamma-1)^n \in \mathscr{H}$, we shall set $H' := \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)c_{n+1}(H)}{\log_p \chi_{\mathrm{cyc}}(\gamma)}(\gamma-1)^n \in \mathscr{H}$. Notice then that $\mathbb{1}(H') = c_1(H)/\log_p \chi_{\mathrm{cyc}}(\gamma)$ does not depend on the choice of γ.

We shall equip $\mathcal{D}_h(\Gamma)$ $(0 \le h \le \infty)$ and $\mathscr{H}$ with a Λ-module and Galois module structure via the compositum of the maps

$$\Lambda \hookrightarrow \Lambda[1/p] = \mathcal{D}_0(\Gamma) \hookrightarrow \mathcal{D}(\Gamma) \overset{\text{Amice}}{\hookrightarrow} \mathscr{H}\,.$$

We set $\mathscr{H}_{\mathscr{A}} := \mathscr{H}\widehat{\otimes}\mathscr{A}$ given a $\mathbb{Q}_p$-affinoid $\mathscr{A}$. We let $\mathscr{H}^{\iota} := \mathscr{H} \otimes_{\Lambda} \Lambda^{\iota}$ and similarly define $\mathscr{H}^{\iota}_{\mathscr{A}}$.

2.1. **Modular curves, Hecke correspondences and the weight space.** For each non-negative integer $s \in \mathbb{Z}_{\geq 2}$, we let Y_s denote the affine modular curve of level $\Gamma_0(N) \cap \Gamma_1(p^s)$. It parametrizes triples (E, C, ϖ) where E is an elliptic curve, C is a cyclic group of E of order N and ϖ is a point of order p^s. We let X_s denote its compactification and $J_s := \mathrm{Jac}(X_s)$.

For each s, we let $\mathfrak{H}_s \subset \mathrm{End}(J_s)$ denote $\mathbb{Z}_p$-the algebra generated by all Hecke operators $\{T_\ell\}_{\ell \nmid Np}$ together with $\{U_\ell\}_{\ell \mid Np}$ and the diamond operators $\{\langle m \rangle : m \in (\mathbb{Z}/p^s\mathbb{Z})^\times\}$.

We set $\Lambda_{\mathrm{wt}} := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. For $z \in \mathbb{Z}_p^\times$, we let $[z] \in \Lambda_{\mathrm{wt}}$ denote the group-like element. The Hecke algebra $\mathfrak{H}_s$ comes equipped with a Λ_{wt}-module structure via $[z] \mapsto \langle z \rangle$. We let $\mathfrak{m}_s$ denote the maximal ideal of $\mathfrak{H}_s$ that is determined by the residual representation $\overline{\rho}_f$ associated to our fixed eigenform f. When there is no risk of confusion, we shall abbreviate $\mathfrak{m} := \mathfrak{m}_s$.

Following [How07], we define the critical weight character $\Theta : \Gamma_{\mathrm{cyc}} \to \Lambda_{\mathrm{wt}}$ (centered at weight k) by setting

$$\Theta(\sigma) := \omega^{\frac{k}{2}-1}(\sigma)[\langle \chi_{\mathrm{cyc}} \rangle^{1/2}(\sigma)]$$

for $\sigma \in \Gamma_{\mathrm{cyc}}$, where $\omega : \Gamma_{\mathrm{cyc}} \to \mathbb{Z}_p^\times$ is the Teichmüller character. We let $\Lambda_{\mathrm{wt}}^\dagger$ denote Λ_{wt} as a module over itself, but allowing $G_{\mathbb{Q}}$ act via the character Θ^{-1}. Let $\xi \in \Lambda_{\mathrm{wt}}^\dagger$ denote the element that corresponds to $1 \in \Lambda_{\mathrm{wt}}$.

For any $\mathfrak{H}_s$-module M on which $G_{\mathbb{Q}}$ acts, we shall write

$$M^\dagger := M \otimes_{\Lambda_{\mathrm{wt}}} \Lambda_{\mathrm{wt}}^\dagger$$

which we equip with the diagonal $G_{\mathbb{Q}}$-action. Here the tensor product is over Λ_{wt} and its action on M is given via the morphism $\Lambda_{\mathrm{wt}} \to \mathfrak{H}_s$ (the diamond action).

2.2. **The Coleman family $(\underline{\mathbf{f}}, \beta)$.** We fix an isomorphism $e_{k-2}\Lambda_{\mathrm{wt}} \cong \mathbb{Z}_p[[w]]$ and let $\mathscr{W} := \mathrm{Sp}\,\mathbb{Z}_p[[w]]$ denote the weight space and let $U = \overline{B}(k, p^{-r}) \subset \mathscr{W}$ denote the closed disk about k of radius p^{-r} for some positive integer r. We let $\mathcal{O}(U)$ denote the ring of analytic functions on the affinoid U; the ring $\mathcal{O}(U)$ is isomorphic to the Tate algebra $\mathscr{A} = E\,\langle\langle w/p^r \rangle\rangle$.

For each $\kappa \in k + p^{r-1}\mathbb{Z}_p$, we shall denote by ψ_κ the morphism

$$\begin{aligned} \psi_\kappa : \mathscr{A} &\longrightarrow E \\ w &\longmapsto (1+p)^{\kappa-k} - 1\,. \end{aligned}$$

Consider the sequence $I = \{\kappa \in \mathbb{Z}_{\geq 2} \mid \kappa \equiv k \pmod{(p-1)p^{r-1}}\}$ of integers and let

$$\underline{\mathbf{f}} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathscr{A}[[q]]$$

denote a p-adic family of cuspidal eigenforms passing through f^β, in the sense of Coleman [Col97]. This means that for every point $\kappa \in I$, the formal expression

$$\underline{\mathbf{f}}(\kappa) := \sum_{n=1}^{\infty} \psi_\kappa(\mathbf{a}_n) q^n$$

is the q-expansion of a cuspidal eigenform of level $\Gamma_0(Np)$ and weight k, with the additional property that $\underline{\mathbf{f}}(k) = f$. Let us denote by $\beta = \mathbf{a}_p$ for the U_p-eigenvalue for its action on $\underline{\mathbf{f}}$, so that we have $\beta(k) = \beta$. By shrinking the closed ball U if necessary and using [Col97, Corollary B5.7.1], we may (and we will henceforth) assume that $\underline{\mathbf{f}}$ is a family of constant slope $k-1$ (in particular, $\underline{\mathbf{f}}$ specialises to a classical form of weight w and slope $k-1$ at any integer weight $w > k$ lying in U).

Let $W_{\underline{\mathbf{f}}}$ denote the big Galois representation associated to the family $\underline{\mathbf{f}}$ with coefficients in $\mathscr{A} = \mathcal{O}(U)$. We define its twist $V_{\underline{\mathbf{f}}} := W_{\underline{\mathbf{f}}} \otimes_{\Lambda_{\mathrm{wt}}} \Lambda_{\mathrm{wt}}^\dagger$. We recall that $W_{\underline{\mathbf{f}}}$ comes equipped with a Λ_{wt}-module structure via the diamond action. Note then that $V_{\underline{\mathbf{f}}}$ is self-dual in the sense that we have a $G_{\mathbb{Q},\Sigma}$-equivariant symplectic pairing (that we denote by $\langle\, ,\, \rangle_{Np^\infty}$)

$$\langle\, ,\, \rangle_{Np^\infty} \,:\, V_{\underline{\mathbf{f}}} \times V_{\underline{\mathbf{f}}} \longrightarrow \mathscr{A}(1)\,. \tag{1}$$

3. Selmer complexes and p-adic heights in families

3.1. **Cohomology of $(\varphi, \Gamma_{\mathbf{cyc}})$-modules.** In this subsection, we shall review the cohomology of $(\varphi, \Gamma_{\mathrm{cyc}})$-modules. Fix a topological generator γ of Γ. Recall that $\mathscr{A}$ stands for the affinoid algebra over E and $\mathcal{R}_{\mathscr{A}}$ for the relative Robba ring over $\mathscr{A}$. For any $(\varphi, \Gamma_{\mathrm{cyc}})$-module $\mathbb{D}$ over $\mathcal{R}_{\mathscr{A}}$ consider the Fontaine–Herr complex

$$C^\bullet_{\varphi,\gamma}(\mathbb{D}) \;:\; \mathbb{D}^\Delta \xrightarrow{d_0} \mathbb{D}^\Delta \oplus \mathbb{D}^\Delta \xrightarrow{d_1} \mathbb{D}^\Delta,$$

where $d_0(x) = ((\varphi-1)(x), (\gamma-1)x)$ and $d_1(y,z) = (\gamma-1)(y) - (\varphi-1)(z)$ (for further details and properties, see [Her98, Liu08, KPX14]). We define

$$H^i(\mathbb{D}) := H^i(C^\bullet_{\varphi,\gamma}(\mathbb{D})).$$

It follows from [Liu08, Theorem 0.2] and [KPX14, Theorem 4.4.2] that $H^i(\mathbb{D})$ is a finitely generated $\mathscr{A}$-module for $i = 0, 1, 2$.

In the particular case when $\mathbb{D} = \mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V_{\underline{\mathbf{f}}})$, it follows by [Liu08, Theorem 0.1] and [Pot13, Theorem 2.8]) that there exist canonical (up to the choice of γ) and functorial isomorphisms

$$H^i(\mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V_{\underline{\mathbf{f}}})) \simeq H^i(\mathbb{Q}_p, V_{\underline{\mathbf{f}}}). \tag{2}$$

for each i. The following proposition is due to Benois [Ben14, Proposition 2.4.2] and refines the isomorphism (2). Set

$$K_p^\bullet(V_{\underline{\mathbf{f}}}) := \mathrm{Tot}\left(C^\bullet\left(G_p, V_{\underline{\mathbf{f}}} \otimes_{\mathscr{A}} \widetilde{\mathbb{B}}^\dagger_{\mathrm{rig},\mathscr{A}}\right) \xrightarrow{\varphi-1} C^\bullet\left(G_p, V_{\underline{\mathbf{f}}} \otimes_{\mathscr{A}} \widetilde{\mathbb{B}}^\dagger_{\mathrm{rig},\mathscr{A}}\right)\right),$$

where $\widetilde{\mathbb{B}}^\dagger_{\mathrm{rig},\mathscr{A}}$ is the ring of p-adic periods introduced by Berger in [Ber02].

Proposition 3.1.1 (Benois)**.** *We have a diagram*

$$\begin{array}{ccc} C^\bullet(G_p, V_{\underline{\mathbf{f}}}) & \xrightarrow[\simeq]{\xi} & K_p^\bullet(V_{\underline{\mathbf{f}}}), \\ & & \uparrow{\scriptstyle \eta\,|\,\simeq} \\ & & C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{\mathrm{rig}}(V_{\underline{\mathbf{f}}})) \end{array}$$

where the maps η and ξ are both quasi-isomorphisms.

3.2. **Selmer complexes.**

3.2.1. *Local conditions at primes above p.* A result of Liu [Liu08, Theorem 0.3.4] shows that the $(\varphi, \Gamma_{\mathrm{cyc}})$-module $\mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V_{\underline{\mathbf{f}}})$ admits a triangulation over $\mathscr{A}$. In more precise terms, the module $\mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V_{\mathbf{f}})$ sits in an exact sequence

$$0 \to \mathbb{D}_\beta \to \mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V_{\mathbf{f}}) \to \widetilde{\mathbb{D}}_\beta \to 0, \tag{3}$$

where both $\mathbb{D}_\beta$ and $\widetilde{\mathbb{D}}_\beta$ are $(\varphi, \Gamma_{\mathrm{cyc}})$-modules of rank 1.

Recall that we have assumed $p = \mathfrak{p}\mathfrak{p}^c$ splits, so that $K_{\mathfrak{q}} = \mathbb{Q}_p$ for each $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^c\}$. We define $U^+_{\mathfrak{q}}(V_{\underline{\mathbf{f}}}, \mathbb{D}_\beta) := C^\bullet_{\varphi,\gamma}(\mathbb{D}_\beta)$. On composing the quasi-isomorphism η of Proposition 3.1.1 with the canonical morphism $U^+_{\mathfrak{q}}(V_{\underline{\mathbf{f}}}, \mathbb{D}_\beta) \to C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{\mathrm{rig},\mathscr{A}}(V))$, we obtain a map

$$i^+_{\mathfrak{q}} \,:\, U^+_{\mathfrak{q}}(V_{\underline{\mathbf{f}}}, \mathbb{D}_\beta) \longrightarrow K^\bullet_{\mathfrak{q}}(V_{\underline{\mathbf{f}}})$$

where $K^\bullet_{\mathfrak{q}}(V_{\underline{\mathbf{f}}}) := \mathrm{Tot}\left(C^\bullet\left(G_{\mathfrak{q}}, V_{\underline{\mathbf{f}}} \otimes_{\mathscr{A}} \widetilde{\mathbb{B}}^\dagger_{\mathrm{rig},\mathscr{A}}\right) \xrightarrow{\varphi-1} C^\bullet\left(G_{\mathfrak{q}}, V_{\underline{\mathbf{f}}} \otimes_{\mathscr{A}} \widetilde{\mathbb{B}}^\dagger_{\mathrm{rig},\mathscr{A}}\right)\right)$ as above.

3.2.2. *Local conditions away from p.* For each non-archimedean prime $\lambda \in \Sigma \setminus \{\mathfrak{p}, \mathfrak{p}^c\}$ of K, we define the complex

$$U^+_\lambda(V) = \left[V_{\underline{\mathbf{f}}}^{I_\lambda} \xrightarrow{\mathrm{Fr}_\lambda - 1} V_{\underline{\mathbf{f}}}^{I_\lambda}\right],$$

which is concentrated in degrees 0 and 1 and where Fr_λ denotes the geometric Frobenius. We define

$$i^+_\lambda \,:\, U^+_\lambda(V_{\underline{\mathbf{f}}}) \longrightarrow C^\bullet(G_\lambda, V_{\underline{\mathbf{f}}})$$

by setting

$$\begin{aligned} i^+_\lambda(x) &= x && \text{in degree } 0, \\ i^+_\lambda(x)(\mathrm{Fr}_\lambda) &= x && \text{in degree } 1. \end{aligned}$$

In order to have a uniform notation for all primes in Σ we set $K^\bullet_\lambda(V_{\underline{\mathbf{f}}}) := C^\bullet(G_\lambda, V_{\underline{\mathbf{f}}})$ and $U^+_\lambda(V_{\underline{\mathbf{f}}}, \mathbb{D}_\beta) := U^+_\lambda(V_{\underline{\mathbf{f}}})$ for a non-archimedean prime $\lambda \in \Sigma \setminus \{\mathfrak{p}, \mathfrak{p}^c\}$. Since we assume $p > 2$, we may safely ignore the archimedean places.

3.2.3. *The Selmer complex.* We define the complexes $K^\bullet_\Sigma(V_{\underline{\mathbf{f}}}) := \bigoplus_{\lambda\in\Sigma} K^\bullet_\lambda(V_{\underline{\mathbf{f}}})$ and $U^+_\Sigma(V_{\underline{\mathbf{f}}},\mathbb{D}_\beta) := \bigoplus_{\lambda\in\Sigma} U^+_\lambda(V_{\underline{\mathbf{f}}},\mathbb{D}_\beta)$. Observe that we have a diagram

$$\begin{tikzcd} C^\bullet(G_{K,\Sigma},V_{\underline{\mathbf{f}}}) \arrow[r, "\mathrm{res}_\Sigma"] & K^\bullet_\Sigma(V_{\underline{\mathbf{f}}}) \\ & U^+_\Sigma(V_{\underline{\mathbf{f}}},\mathbb{D}_\beta), \arrow[u, "i^+_\Sigma"] \end{tikzcd}$$

where $i^+_\Sigma = (i^+_\lambda)_{\lambda\in\Sigma}$ and res_Σ denotes the localization map.

Definition 3.2.1. *The Selmer complex associated to these data is defined as*

$$S^\bullet(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) = \mathrm{cone}\left(C^\bullet(G_{K,\Sigma},V_{\underline{\mathbf{f}}}) \oplus U^+_\Sigma(V_{\underline{\mathbf{f}}},\mathbb{D}_\beta) \xrightarrow{\mathrm{res}_\Sigma - i^+_\Sigma} K^\bullet_\Sigma(V_{\underline{\mathbf{f}}})\right)[-1].$$

Definition 3.2.2. *We denote by $\mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)$ the class of $S^\bullet(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)$ in the derived category of $\mathscr{A}$-modules and denote by*

$$H^i(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) := \mathbf{R}^i\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta).$$

its cohomology.

3.3. **$\mathscr{A}$-adic cyclotomic height pairings.** We provide in this section an overview of the construction of p-adic heights for p-adic representations over the affinoid algebra $\mathscr{A}$, following [Ben14]. We retain our previous notation and conventions.

Let $J_{\mathscr{A}}$ denote the kernel of the augmentation map

$$\mathscr{H}\,\widehat{\otimes}\,\mathscr{A} =: \mathscr{R}_{+,\mathscr{A}} \to \mathscr{A}$$

which is induced by $\gamma \mapsto 1$. Note that $J_{\mathscr{A}} = (\gamma-1)\mathscr{R}_{+,\mathscr{A}}$ and $J_{\mathscr{A}}/J^2_{\mathscr{A}} \simeq \mathscr{A}$ as $\mathscr{A}$-modules. The exact sequence

$$0 \to V_{\underline{\mathbf{f}}} \otimes J_{\mathscr{A}}/J^2_{\mathscr{A}} \longrightarrow V_{\underline{\mathbf{f}}} \otimes R_{+,\mathscr{A}}/J^2_{\mathscr{A}} \longrightarrow V_{\underline{\mathbf{f}}} \longrightarrow 0$$

and the functorial behaviour of Selmer complexes under base change induces the Bockstein morphism

$$\beta^{\mathrm{cyc}}_{V_{\underline{\mathbf{f}}},\mathbb{D}_\beta} : \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \longrightarrow \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)[1] \otimes_{\mathscr{A}} J_{\mathscr{A}}/J^2_{\mathscr{A}}.$$

Definition 3.3.1. *The p-adic height pairing associated to the Coleman family $(\underline{\mathbf{f}},\beta)$ is defined as the morphism*

$$\mathfrak{h}_{\underline{\mathbf{f}},\beta} : \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \otimes^{\mathbb{L}}_{\mathscr{A}} \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \xrightarrow{\beta^{\mathrm{cyc}}_{V_{\underline{\mathbf{f}}},\mathbb{D}_\beta}\otimes \mathrm{id}}$$
$$\left(\mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)[1] \otimes J_{\mathscr{A}}/J^2_{\mathscr{A}}\right) \otimes^{\mathbb{L}}_{\mathscr{A}} \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \xrightarrow{\cup} J_{\mathscr{A}}/J^2_{\mathscr{A}}[-2]$$

where $\cup$ is the cup-product pairing

$$\mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \otimes^{\mathbb{L}}_{\mathscr{A}} \mathbf{R}\Gamma(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \xrightarrow{\cup} \mathscr{A}[-3]$$

which is induced from the $G_{K,\Sigma}$-equivariant symplectic pairing $\langle\,,\,\rangle_{Np^\infty}$ of (1).

In the level of cohomology, $h_{\underline{\mathbf{f}},\beta}$ induces a pairing

$$\mathfrak{h}^{1,1}_{\underline{\mathbf{f}},\beta} : H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \otimes_{\mathscr{A}} H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \longrightarrow J_{\mathscr{A}}/J^2_{\mathscr{A}}.$$

Proposition 3.3.2. *The $\mathscr{A}$-adic height pairing $\mathfrak{h}^{1,1}_{\underline{\mathbf{f}},\beta}$ is symmetric.*

Proof. This is a direct consequence of [Ben14, Theorem I] and the fact that the pairing $\langle\, , \,\rangle_{Np^\infty}$ is symplectic. □

The map $\gamma - 1 \pmod{J^2_{\mathscr{A}}} \mapsto \log \chi_{\mathrm{cyc}}(\gamma)$ induces an isomorphism

$$\partial_{\mathrm{cyc}} : J_{\mathscr{A}}/J^2_{\mathscr{A}} \xrightarrow{\sim} \mathscr{A}\,.$$

We define the $\mathscr{A}$-valued height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$ by setting

$$\mathfrak{h}_{\underline{\mathbf{f}},K} := \partial_{\mathrm{cyc}} \circ \mathfrak{h}^{1,1}_{\underline{\mathbf{f}},\beta}\,.$$

3.4. **Specializations and comparison with Nekovář's heights.** Shrinking U if necessary, we shall assume that $2k \notin I$. Throughout this subsection, we fix an integer $\kappa \in I$ with $\kappa \geq k$ and set

$$g := \underline{\mathbf{f}}(\kappa) \in S_\kappa(\Gamma_0(Np)) \quad \text{and} \quad b = \beta(\kappa).$$

The Galois representation $V_{\underline{\mathbf{f}}} \otimes_{\mathscr{A},\psi_\kappa} E$ is the central-critical twist V_g of Deligne's representation W_g associated to the cuspidal eigenform g.

Lemma 3.4.1. *The eigenform g is non-θ-critical and old at p.*

Proof. If $\kappa > k$, the eigenform for g is not critical since in this case we have $v_p(b) = k-1 < \kappa - 1$. If $\kappa = k$, then $g = f^\beta$ is non-θ-critical by assumption.

If g were new at p, we would have $k - 1 = v_p(b) = \kappa/2 - 1$ and thus $\kappa = 2k$, contradicting the choice of I. □

Corollary 3.4.2. *The Galois representation V_g is crystalline at p.*

Definition 3.4.3. *We let $g_\circ \in S_\kappa(\Gamma_0(N))$ denote the newform such that $g = g_\circ^b$ is the p-stabilization of $g_\circ$.*

Consider the Bloch-Kato Selmer group $H^1_{\mathrm{f}}(K, V_g)$. It comes equipped with Nekovář's p-adic height pairing

$$h^{\mathrm{Nek}}_{b,K} : H^1_{\mathrm{f}}(K, V_g) \otimes H^1_{\mathrm{f}}(K, V_g) \longrightarrow E.$$

The height pairing $h^{\mathrm{Nek}}_{b,K}$ is associated to the Hodge-splitting

$$\mathrm{D}_{\mathrm{cris}}(V_g) = \mathrm{D}_b \oplus \mathrm{Fil}^0\mathrm{D}_{\mathrm{cris}}(V_g)$$

together with the symplectic pairing

$$\langle\, , \,\rangle_N : V_g \otimes V_g \longrightarrow E(1)$$

that is induced from the Poincaré duality for the étale cohomology of the modular curve $X_0(N)$, where $V_g = V_{g_\circ}$ appears as a direct summand.

Our goal in this subsection is to compare these objects to those obtained by specializing the $\mathscr{A}$-adic objects we have defined in the previous section.

Definition 3.4.4.

i) *We let W_{Np} denote the Atkin-Leher operator of level Np and let $\langle\, , \,\rangle_{Np}$ denote the Poincaré duality pairing on the cohomology of the modular curve of level $\Gamma_0(N) \cap \Gamma_1(p)$.*

ii) *Realizing* V_g *as the* g*-isotypical (with respect to the Hecke operators* T_ℓ *for* $\ell \nmid Np$ *and operators* U_ℓ *for* $\ell \mid Np$*) direct summand in the cohomology of the modular curve of level* $\Gamma_0(N) \cap \Gamma_1(p)$*, we define*

$$\langle\, ,\rangle'_{Np} \,:\, V_g \otimes V_g \longrightarrow E(1)$$

by setting

$$\langle x, y\rangle'_{Np} := \langle x, W_{Np}\, y\rangle_{Np}$$

and refer to it as the p*-stabilized Poincaré duality pairing on* V_g*.*

iii) *We let*

$$\mathrm{Pr}_b^* : V_g \xrightarrow{\sim} V_g$$

denote the natural isomorphism appearing in [KLZ17, Proposition 10.1.1/1][8]*. Here* V_g *on the left is the* g*-isotypical direct summand in the cohomology of* $X_0(N)$ *(with respect to the Hecke operators* T_ℓ *for* $\ell \nmid N$ *and operators* U_ℓ *for* $\ell \mid N$*), whereas* V_g *on the right is the* g*-isotypical direct summand in the cohomology of the modular curve of level* $\Gamma_0(N) \cap \Gamma_1(p)$ *(with respect to the Hecke operators* T_ℓ *for* $\ell \nmid Np$ *and operators* U_ℓ *for* $\ell \mid Np$*).*

We are grateful to D. Loeffler for bringing the following observation to our attention.

Proposition 3.4.5.

i) $\langle \psi_\kappa\, x, \psi_\kappa\, y\rangle'_{Np} = \psi_\kappa \circ \langle x, y\rangle_{Np^\infty}$.

ii) *We have*

$$\langle \mathrm{Pr}_b^*\, x \,,\, \mathrm{Pr}_b^*\, y\rangle'_{Np} = b\, \lambda_N(g_\circ)\, \mathcal{E}(g)\mathcal{E}^*(g)\, \langle x, y\rangle_N,$$

where $\lambda_N(g_\circ)$ *is the Atkin-Lehner pseudo-eigenvalue of* $g_\circ$*,* $\mathcal{E}(g) = \left(1 - \frac{p^{\kappa-2}}{b^2}\right)$ *and* $\mathcal{E}^*(g) = \left(1 - \frac{p^{\kappa-1}}{b^2}\right)$*.*

Proof. The first assertion is well-known; c.f. Proposition 4.4.8 and Theorem 4.6.6 of [LZ16]. For the second, we note that

$$\begin{aligned}\langle \mathrm{Pr}_b^*\, x \,,\, \mathrm{Pr}_b^*\, y\rangle'_{Np} &= \langle \mathrm{Pr}_b^*\, x \,,\, W_{Np}\, \mathrm{Pr}_b^*\, y\rangle_{Np} \\ &= \langle x \,,\, (\mathrm{Pr}_b)_*\, W_{Np}\, \mathrm{Pr}_b^*\, y\rangle_N \\ &= b\, \lambda_N(g_\circ)\, \mathcal{E}(g)\mathcal{E}^*(g)\, \langle x, y\rangle_N\end{aligned}$$

where the first and second equalities follows from definitions, whereas the third is a consequence of the discussion in the final paragraph of the proof of Proposition 10.1.1 of [KLZ17]. □

Since the g is non-θ-critical (Lemma 3.4.1), the triangulation (3) gives rise to a saturated triangulation

$$0 \longrightarrow \mathbb{D}_b \longrightarrow \mathbb{D}^\dagger_{\mathrm{rig}}(V_g) \longrightarrow \widetilde{\mathbb{D}}_b \longrightarrow 0$$

of the $(\varphi, \Gamma_{\mathrm{cyc}})$-module $\mathbb{D}^\dagger_{\mathrm{rig}}(V_g)$ by base change, where $\mathbb{D}_b := \mathbb{D}_\beta \otimes_{\mathscr{A},\psi_\kappa} E$ and $\widetilde{\mathbb{D}}_b := \widetilde{\mathbb{D}}_\beta \otimes_{\mathscr{A},\psi_\kappa} E$. With this data at hand, one may proceed precisely as in Section 3.2.3 to define a Selmer complex $S^\bullet(V_{g/K}, \mathbb{D}_b)$ in the category of E-vector

[8] This map would have been denoted by $(\mathrm{Pr}^b)^*$ in op. cit.

spaces. We let $\mathbf{R}\Gamma(V_{g_{/K}},\mathbb{D}_b)$ denote the corresponding object in the derived category and $H^i(V_{g_{/K}},\mathbb{D}_b)$ denote its cohomology.

The general formalism to construct p-adic heights we outlined in Section 3.3 (where we utilize the symplectic pairing

$$\langle\,,\,\rangle'_{Np}\,:\,V_g\otimes V_g\longrightarrow E(1)$$

given in Definition 3.4.4 to determine an isomorphism $V_g^*(1)\stackrel{\sim}{\to}V_g$) also equips us with an E-valued height pairing

$$\mathfrak{h}_{g,b,K}\,:\,H^1(V_{g_{/K}},\mathbb{D}_b)\otimes H^1(V_{g_{/K}},\mathbb{D}_b)\longrightarrow E\,.$$

Lemma 3.4.6.

i) *We have a natural morphism (which we shall denote by ψ_κ, by slight abuse)*

$$\psi_\kappa:\,H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)\otimes_{\mathscr{A},\psi_\kappa}E\longrightarrow H^1(V_{g_{/K}},\mathbb{D}_b)$$

of E-vector spaces, which is an isomorphism for all but finitely many choices of g.

ii) *The following diagram commutes:*

$$\begin{tikzcd}
H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \arrow[d, "\psi_\kappa"'] & \otimes_{\mathscr{A}} & H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta) \arrow[r, "\mathfrak{h}_{\underline{\mathbf{f}},K}"] \arrow[d, "\psi_\kappa"'] & \mathscr{A} \arrow[d, "\psi_\kappa"'] \\
H^1(V_{g_{/K}},\mathbb{D}_b) & \otimes_E & H^1(V_{g_{/K}},\mathbb{D}_b) \arrow[r, "\mathfrak{h}_{g,b,K}"'] & E
\end{tikzcd}$$

Proof. Let $\wp_\kappa:=\ker(\psi_\kappa)$ be the prime of $\mathscr{A}$ corresponding to g. Notice then that

$$H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)\otimes_{\mathscr{A},\psi_\kappa}E=H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)/\wp_\kappa H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)$$

and the general base change principles for Selmer complexes (c.f. [Pot13, Section 1]) shows that the sequence

$$0\longrightarrow H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)/\wp_\kappa H^1(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)\longrightarrow H^1(V_{g_{/K}},\mathbb{D}_b)\longrightarrow H^2(V_{\underline{\mathbf{f}}_{/K}},\mathbb{D}_\beta)[\wp_\kappa]$$

of E-vector spaces is exact. The first assertion now follows. The second follows easily from definitions. □

Proposition 3.4.7. *There is a natural isomorphism*

$$H^1(V_{g_{/K}},\mathbb{D}_b)\stackrel{\sim}{\longrightarrow}H^1_{\mathrm{f}}(K,V_g)\,.$$

Moreover, the height pairing $\mathfrak{h}_{g,b,K}$ coincides with $h^{\mathrm{Nek}}_{b,K}/b\,\lambda_N(g_\circ)\,\mathcal{E}(g)\mathcal{E}^*(g)$.

Proof. The proof of the first assertion reduces to [Ben14, Theorem III] once we verify

(i) $\mathrm{D}_{\mathrm{cris}}(V_g)^{\varphi=1}=0$,
(ii) $H^0(\widetilde{\mathbb{D}}_b)=0$.

Assume first $\kappa\neq k$ (so that $g\neq f^\beta$). Let $g_\circ$ be as in Definition 3.4.3. The roots of the Hecke polynomial for $g_\circ$ at p could not be the pair $\{1,p^{\kappa-1}\}$, as otherwise we would have $\kappa-1=v_p(b)=k-1$. This verifies both conditions in this case.

When $\kappa = k$ and $g = f^\beta$, both conditions follow as a consequence of the Ramanujan-Petersson conjecture for f (as proved by Deligne), according to which the roots of the Hecke polynomial of f at p could not be the pair $\{1, p^{k-1}\}$.

The assertion concerning the comparison of two p-adic heights follows from [Ben14, Theorem 11] together with Proposition 3.4.5. (We find it instructive to compare Benois' result to [Nek06, Theorem 11.4.6] in the ordinary case.) $\square$

The following commutative diagram summarizes the discussion in this subsection:

(4)
$$\begin{tikzcd}
H^1(V_{\underline{\mathbf{f}}/K}, \mathbb{D}_{\beta}) \arrow[d, "\psi_\kappa"'] & \otimes_{\mathscr{A}} & H^1(V_{\underline{\mathbf{f}}/K}, \mathbb{D}_{\beta}) \arrow[d, "\psi_\kappa"'] \arrow[r, "\mathfrak{h}_{\underline{\mathbf{f}},K}"] & \mathscr{A} \arrow[d, "\psi_\kappa"'] \\
H^1(V_{g/K}, \mathbb{D}_b) & \otimes_E & H^1(V_{g/K}, \mathbb{D}_b) \arrow[r, "\mathfrak{h}_{g,b,K}"'] & E \arrow[d, "b\,\lambda_N(g_\circ)\,\mathcal{E}(g)\mathcal{E}^*(g)"] \\
H^1_{\mathrm{f}}(K, V_g) \arrow[u, "\cong", "\mathrm{Pr}_b^*"'] & \otimes_E & H^1_{\mathrm{f}}(K, V_g) \arrow[u, "\cong", "\mathrm{Pr}_b^*"'] \arrow[r, "h^{\mathrm{Nek}}_{b,K}"'] & E
\end{tikzcd}$$

3.5. **Universal Heegner points.** In this subsection, we shall introduce elements in the Selmer groups on which we shall calculate the $\mathscr{A}$-adic height $\mathfrak{h}_{\underline{\mathbf{f}},K}$.

3.5.1. *Heegner cycles.* We recall the definition of Heegner cycles on Kuga-Sato varieties, following the discussion in [Nek95]. Recall that we have fixed an imaginary quadratic field K such that all primes dividing the tame level Np splits completely in $K/\mathbb{Q}$. Let $g \in S_\kappa(\Gamma_0(N))$ be a cuspidal eigenform of weight $\kappa > 2$.

Let $Y(N)$ denote the modular curve over $\mathbb{Q}$ which is the moduli of elliptic curves with full level N structure and we let $\mathfrak{j} : Y(N) \to X(N)$ denote its non-singular compactification. Since we assume $N \geq 3$, there is a universal generalized elliptic curve $E \to X(N)$ that restricts to the universal elliptic curve $f : E \to Y(N)$. The $(\kappa-2)$-fold fibre product of E with itself over $Y(N)$ has a canonical non-singular compactification W described in detail in [Del71, Sch90]. We have natural maps

(5) $$H^{\kappa-1}_{\text{ét}}(W \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)(\kappa/2) \to H^1_{\text{ét}}(X(N) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathfrak{j}_*\mathrm{Sym}^{\kappa-2}(R^1 f_* \mathbb{Q}_p))(\kappa/2) \to V_g\,.$$

Scholl defines a projector ε (where his w corresponds to our $\kappa - 2$) and proves that there is a canonical isomorphism

$$H^1_{\text{ét}}(X(N) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathfrak{j}_*\mathrm{Sym}^{\kappa-2}(R^1 f_* \mathbb{Q}_p)) \xrightarrow{\sim} \varepsilon H^{\kappa-1}_{\text{ét}}(W \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)\,.$$

We finally define

$$\mathscr{B} := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \Big/ \{\pm 1\} \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \big/ \{\pm 1\}$$

and the idempotent $\varepsilon_{\mathscr{B}} := \frac{1}{|\mathscr{B}|} \sum_{g \in \mathscr{B}} g$ (which acts on the modular curves $Y(N)$ and $X(N)$).

Definition 3.5.1. *We let $\mathfrak{N}$ be an ideal of $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. For an arbitrarily chosen ideal $\mathcal{A} \subset \mathcal{O}$, consider the isogeny $\mathbb{C}/\mathcal{A} \to \mathbb{C}/\mathcal{A}\mathfrak{N}^{-1}$. It represents* *the Heegner point $y = y_{\mathcal{A}}$ on $Y_0(N)(\mathbb{C})$. It is defined over the Hilbert class field H of K.*

Choose any point $\widetilde{y} \in Y(N) \times_{\mathbb{Q}} H$ over the Heegner point y (viewed as a closed point of $Y_0(N) \times_{\mathbb{Q}} H$). The fiber $E_{\widetilde{y}}$ is a CM elliptic curve defined over H whose endomorphism ring is isomorphic to $\mathcal{O}$. We let

$$\Gamma_{\sqrt{D_K}} \subset E_{\widetilde{y}} \times E_{\widetilde{y}}$$

denote the graph of $\sqrt{D_K} \in \mathcal{O}$ (fix any one of the two square-roots)

Definition 3.5.2. *We let*

$$Y := \underbrace{\Gamma_{\sqrt{D_K}} \times \cdots \times \Gamma_{\sqrt{D_K}}}_{\kappa/2-1 \text{ times}} \subset \underbrace{E_{\widetilde{y}} \times \cdots \times E_{\widetilde{y}}}_{\kappa-2 \text{ times}} = (W \times_{\mathbb{Q}} H)_{\widetilde{y}}$$

and call the cycle (with rational coefficients) that is represented by $\varepsilon_{\mathscr{B}}\varepsilon Y$ inside of $\varepsilon_{\mathscr{B}}\varepsilon \mathrm{CH}^{\kappa/2}(W \times_{\mathbb{Q}} H)_0 \otimes \mathbb{Q}$ (which we also denote by the same symbol $\varepsilon_{\mathscr{B}}\varepsilon Y$) the Heegner cycle.

The cohomology class of εY in $H^{\kappa}_{\text{ét}}(W \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)(\kappa/2)$ vanishes, so that one may apply the Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{CH}^{\kappa/2}(W \times_{\mathbb{Q}} H)_0 \otimes \mathbb{Q} \longrightarrow H^1(H, H^{\kappa-1}_{\text{ét}}(W \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)(\kappa/2))$$

on the Heegner cycle $\varepsilon_{\mathscr{B}}\varepsilon Y$.

Definition 3.5.3. *We let $\mathrm{AJ}_g : \mathrm{CH}^{\kappa/2}(W \times_{\mathbb{Q}} H)_0 \otimes \mathbb{Q} \to H^1(H, V_g)$ denote the compositum of the map* (5) *with the Abel-Jacobi map and define the Heegner cycle*

$$z_g := \mathrm{cor}_{H/K}\left(\mathrm{AJ}_g(\varepsilon_{\mathscr{B}}\varepsilon Y)\right) \in H^1(K, V_g)\,.$$

Since $p \nmid N$, all $X(N)$, $X_0(N)$ and W have good reduction at p and it follows from [Nek00, Theorem 3.1(i)] that

$$z_g \in H^1_{\mathrm{f}}(K, V_g)\,.$$

3.5.2. *Heegner cycles in Coleman families.* For a classical weight $\kappa \in I$ and ψ_κ as in Section 2.2, we let $\underline{\mathbf{f}}(\kappa)^\circ \in S_\kappa(\Gamma_0(N))$ denote the newform whose p-stabilization (with respect to $\beta(\kappa)$) is the eigenform $\underline{\mathbf{f}}(\kappa)$.

The following result (construction of a big Heegner point along the Coleman family $\underline{\mathbf{f}}$) is [BL19, Proposition 4.15(iii)] and [JLZ19, Theorem 5.4.1].

Theorem 3.5.4 (Büyükboduk–Lei, Jetchev–Loeffler–Zerbes)**.** *There exists a unique class $\mathscr{Z}_{\underline{\mathbf{f}}} \in H^1(V_{\underline{\mathbf{f}}/K}, \mathbb{D}_\beta)$ that is characterized by the requirement that for any $\kappa \in I$ we have*

$$\psi_\kappa\left(\mathscr{Z}_{\underline{\mathbf{f}}}\right) = u_K^{-1}(2\sqrt{-D_K})^{1-\frac{\kappa}{2}}\left(1 - \frac{p^{\frac{\kappa}{2}-1}}{\beta(\kappa)}\right)^2 z_{\underline{\mathbf{f}}(\kappa)^\circ} \in H^1_{\mathrm{f}}(K, V_{\underline{\mathbf{f}}(\kappa)}),$$

where $u_K = |\mathcal{O}_K^\times|/2$ and $-D_K$ is the discriminant of K.

Remark 3.5.5. Jetchev–Loeffler–Zerbes in [JLZ19] rely on the overconvergent étale cohomology of Andreatta–Iovita–Stevens. The construction of "universal" Heegner cycles in [BL19] exploits the p-adic construction of rational points, a theme first observed by Rubin [Rub92], and dwells on the formula of Bertolini–Darmon–Prasanna which relates the Bloch–Kato logarithms of these cycles to appropriate Rankin–Selberg p-adic L-values. In [BPS19], we will give another construction of "universal" Heegner cycles in the context of Emerton's completed cohomology (on realizing the family $\underline{\mathbf{f}}$ on Emerton's eigensurface).

4. p-adic L-functions over the imaginary quadratic field K

We introduce the needed p-adic L-functions for the arguments in this paper. We first discuss a Rankin-Selberg p-adic L-function defined over our imaginary quadratic field K. We then compare this p-adic L-function to a naïve product of p-adic L-functions defined over $\mathbb{Q}$.

4.1. **Ranking-Selberg p-adic L-functions.** Loeffler and Zerbes in [LZ16] have constructed p-adic L-functions (in 3-variables) associated to families of semi-ordinary Rankin-Selberg products $f_1 \otimes f_2$ of eigenforms, where f_1 runs through a Coleman family and f_2 through a p-ordinary family. (See also [Loe17] where the correct interpolation property is extended from all crystalline points to all critical points.) We shall let f_2 vary in a (suitable branch of the) universal CM family associated to K (which we shall recall below), and thus we may reinterpret this p-adic L-function as a p-adic L-function associated to the base change of $\underline{\mathbf{f}}$ to K.

4.1.1. *CM Hida families.* For a general modulus $\mathfrak{n}$ of K, let $K(\mathfrak{n})$ denote the maximal p-extension contained in the ray class field modulo $\mathfrak{n}$. We set $H^{(p)}_{\mathfrak{n}} := \mathrm{Gal}(K(\mathfrak{n})/K)$. In particular, $K(\mathfrak{p}^\infty) := \cup K(\mathfrak{p}^n)$ is the unique $\mathbb{Z}_p$-extension of K which is unramified outside $\mathfrak{p}$. We let $\Gamma_{\mathfrak{p}} := \varprojlim_n H^{(p)}_{\mathfrak{p}^n}$ denote its Galois group over K. We fix an arbitrary Hecke character ψ_0 of ∞-type $(-1,0)$, conductor $\mathfrak{p}$ and whose associated p-adic Galois character factors through $\Gamma_{\mathfrak{p}}$. Notice then that $\psi_0 \equiv \mathbb{1}_{\mathfrak{p}} \mod \mathfrak{m}_E$, where we have let $\mathbb{1}_{\mathfrak{p}} : (\mathcal{O}/\mathfrak{p})^\times \to \mathcal{O}_E^\times$ denote the trivial character modulo $\mathfrak{p}$.

Remark 4.1.1. If the class number of K is prime to p, then the Hecke character ψ_0 is unique, as the ratio of two such characters would have finite p-power order and conductor dividing $\mathfrak{p}$.

The theta-series

$$\Theta(\psi_0) := \sum_{(\mathfrak{a},\mathfrak{p})=1} \psi_0(\mathfrak{a}) q^{\mathbb{N}\mathfrak{a}} \in S_2(\Gamma_1(|D_K|p), \epsilon_K \omega^{-1})$$

is a newform and it is the weight two specialization (with trivial wild character) of the CM Hida family $\underline{\mathbf{g}}$ with tame level $|D_K|$ and character $\epsilon_K\omega$. The weight one specialization of this CM Hida family with trivial wild character equals the p-ordinary theta-series $\Theta^{\mathrm{ord}}(\mathbb{1}_{\mathfrak{p}}) := \sum_{(\mathfrak{a},\mathfrak{p})=1} q^{\mathbb{N}\mathfrak{a}} \in S_1(\Gamma_1(|D_K|p), \epsilon_K)$ of $\mathbb{1}_{\mathfrak{p}}$.

Remark 4.1.2. One may construct the Hida family $\underline{\mathbf{g}}$ as follows. We let $\mathbf{T}_{|D_K|p}$ denote the Hecke algebra given as in [LLZ15, §4.1] and we define the maximal ideal $\mathcal{I}_{\mathfrak{p}} \subset \mathbf{T}_{|D_K|p}$ as in [LLZ15, Definition 5.1.1]. Note that in order to determine the map $\phi_{\mathfrak{p}}$ that appears in this definition, we use the algebraic Hecke character ψ_0 we have chosen above.

It follows by [LLZ15, Prop. 5.1.2] that $\mathcal{I}_\mathfrak{p}$ is non-Eisenstein, p-ordinary and p-distinguished. By [LLZ15, Theorem 4.3.4], the ideal $\mathcal{I}_\mathfrak{p}$ corresponds uniquely to a p-distinguished maximal ideal $\mathcal{I}$ of the universal ordinary Hecke algebra $\mathbf{T}_{|D_K|p^\infty}$ acting on $H^1_{\rm ord}(Y_1(|D_K|p^\infty))$ (definitions of these objects may be found in [LLZ15, Definition 4.3.1]). The said correspondence is induced from Ohta's control theorem [Oht99, Theorem 1.5.7(iii)], which also attaches to $\mathcal{I}_\mathfrak{p}$ a unique non-Eisenstein, p-ordinary and p-distinguished maximal ideal $\mathcal{I}_{\mathfrak{p}^r}$ of $\mathbf{T}_{|D_K|p^r}$ for each $r \geq 1$ (it is easy to see that it is the kernel of the compositum of the arrows

$$\mathbf{T}_{|D_K|p^r} \xrightarrow{\phi_{\mathfrak{p}^r}} \mathcal{O}_L[H_{\mathfrak{p}^r}] \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_E/\varpi_E,$$

and therefore with its original form given in [LLZ15, Definition 5.1.1]). The ideal $\mathcal{I}$ determines the CM Hida family $\underline{\mathbf{g}}$ alluded to above.

We shall henceforth identify the rigid analytic ball $\mathrm{Sp}\,\mathbb{Z}_p[[\Gamma_\mathfrak{p}]]$ with the weight space for the Hida family $\underline{\mathbf{g}}$. We let $\widetilde{\kappa} \in \mathrm{Sp}\,\mathbb{Z}_p[[\Gamma_\mathfrak{p}]]$ denote the point corresponding to the weight one specialization $\Theta^{\rm ord}(\mathbb{1}_\wp)$.

4.1.2. *The p-adic L-function and interpolation property.* We fix an affinoid neighborhood $\mathscr{Y} \subset \mathrm{Sp}\,\mathbb{Z}_p[[\Gamma_p]]$ and let

$$L_p^{\rm RS}(\underline{\mathbf{f}}, \underline{\mathbf{g}}|_{\mathscr{Y}}) \in \mathscr{O}(\mathscr{Y})\,\widehat{\otimes}\,\mathscr{H}_{\mathscr{A}}$$

denote the 3-variable Rankin-Selberg p-adic L-function of Loeffler and Zerbes [LZ16, Loe17]. Since $\underline{\mathbf{g}}$ is a p-ordinary family, we may choose $\mathscr{Y}$ as large as we like and obtain a p-adic L-function

$$L_p^{\rm RS}(\underline{\mathbf{f}}, \underline{\mathbf{g}}) \in \mathscr{H}(\Gamma_\mathfrak{p})\,\widehat{\otimes}\,\mathscr{H}_{\mathscr{A}}\,.$$

As explained in detail in [BL17a], the p-adic L-function may be thought as a relative p-adic L-function for $\underline{\mathbf{f}}$ over K, interpolating the algebraic parts of the L-values $L(\underline{\mathbf{f}}(\kappa)_{/K}, \Psi, 1)$ where Ψ runs through the algebraic Hecke characters of K with infinity type (a, b) with $0 \leq a \leq b \leq \kappa - 2$.

Definition 4.1.3. *We let $\mathscr{D}^{\rm RS}_{\underline{\mathbf{f}}_{/K}} \in \mathscr{H}_{\mathscr{A}}$ denote the p-adic distribution obtained by specializing $L_p^{\rm RS}(\underline{\mathbf{f}}, \underline{\mathbf{g}})$ to the point $\widetilde{\kappa} \in \mathrm{Sp}\,\mathbb{Z}_p[[\Gamma_p]]$ in the weight space for $\underline{\mathbf{g}}$, corresponding to the weight one specialization $\Theta^{\rm ord}(\mathbb{1}_\wp)$.*

The following interpolation property characterizes the distribution $\mathscr{D}^{\rm RS}_{\underline{\mathbf{f}}_{/K}}$.

Theorem 4.1.4 (Loeffler)**.** *For every $\kappa \in I$, any $j \in \mathbb{Z} \cap [1, \kappa - 1]$ and all Dirichlet characters η of conductor p^r (we allow $r = 0$) we have*

$$\begin{aligned}(\psi_\kappa \otimes \eta\chi_{\rm cyc}^{j-1})(\mathscr{D}^{\rm RS}_{\underline{\mathbf{f}}_{/K}}) = (-1)^{j-1} &\times \frac{\mathcal{V}(\underline{\mathbf{f}}(\kappa)^\circ, \eta, j)^2\, p^{2r(j-\frac{1}{2})} W(\eta \circ \mathbb{N}_{K/\mathbb{Q}})}{\beta(\kappa)^{2r}\mathcal{E}(\underline{\mathbf{f}}(\kappa))\mathcal{E}^*(\underline{\mathbf{f}}(\kappa))} \\ &\times \frac{i^{\kappa-1}N^{2j-\kappa+1}\Gamma(j)^2}{2^{2j+\kappa-1}\pi^{2j}} \times \frac{L(\underline{\mathbf{f}}(\kappa)^\circ_{/K}, \eta^{-1} \circ \mathbb{N}_{K/\mathbb{Q}}, j)}{\langle \underline{\mathbf{f}}(\kappa)^\circ, \underline{\mathbf{f}}(\kappa)^\circ \rangle_N}\end{aligned}$$

where $\mathcal{V}(\underline{\mathbf{f}}(\kappa)^\circ, \eta, j)$ is as in Theorem 4.2.1, $W(\eta \otimes \mathbb{N}_{K/\mathbb{Q}})$ is the root number for the complete Hecke L-series $\Lambda(\eta \circ \mathbb{N}_{K/\mathbb{Q}}, s)$ (c.f. [Nek95, Page 626]) *and finally,*

$$\mathcal{E}(\underline{\mathbf{f}}(\kappa)) := \left(1 - \frac{p^{\kappa-2}}{\beta(\kappa)^2}\right) \quad , \quad \mathcal{E}^*(\underline{\mathbf{f}}(\kappa)) := \left(1 - \frac{p^{\kappa-1}}{\beta(\kappa)^2}\right)\,.$$

Remark 4.1.5. The p-adic L-function (and its interpolation property) recorded in Theorem 4.1.4 is a slight alteration of Loeffler's original formulation in [Loe17]. It can be obtained following the calculations carried out in [BL18] based on Loeffler's work in op. cit.

4.2. **Naïve p-adic L-functions over K.** We now consider a naïve version of a p-adic L-function over K by taking the product of two p-adic L-functions over $\mathbb{Q}$. We begin by recalling two-variable p-adic L-functions over the eigencurve. This construction is due to Glenn Stevens, but first appeared in the literature in [Bel12].

Suppose that $\underline{\mathbf{h}}$ is a Coleman family over a sufficiently small affinoid disc $\mathrm{Sp}(\mathscr{A})$ about a non-θ-critical point g of weight k_0 on the eigencurve (in the sense of Definition 2.12 in [Bel12]) with U_p-eigenvalue α. Let I denote the set of classical weights of forms occurring in $\mathscr{A}$.

Theorem 4.2.1. *There exists a unique p-adic distribution $\mathscr{D}_{\underline{\mathbf{h}}} \in \mathscr{H}_{\mathscr{A}}$ which is characterized by the following interpolation property: For every $\kappa \in I$, any $j \in \mathbb{Z} \cap [1, \kappa - 1]$ and all Dirichlet characters η of conductor $p^r \geq 1$,*

$$(\psi_\kappa \otimes \eta\chi_{\mathrm{cyc}}^{j-1})(\mathscr{D}_{\underline{\mathbf{h}}}) = (-1)^j \Gamma(j) \mathcal{V}(\underline{\mathbf{h}}(\kappa)^\circ, \eta, j) \tau(\eta) \frac{p^{(j-1)r}}{\alpha(\kappa)^r} \frac{L(\underline{\mathbf{h}}(\kappa)^\circ, \eta^{-1}, j)}{(2\pi i)^j \Omega^{\pm}_{\underline{\mathbf{h}}(\kappa)}} C^{\pm}_{\underline{\mathbf{h}}(\kappa)}$$

where,

- *$\tau(\eta)$ is the Gauss sum (normalized to have norm $p^{r/2}$),*
- *$\mathcal{V}(\underline{\mathbf{h}}(\kappa)^\circ, \eta, j) = \left(1 - p^{j-1}\eta(p)/\alpha(\kappa)\right)\left(1 - p^{\kappa-1-j}\eta^{-1}(p)/\alpha(\kappa)\right)$,*
- *$\Omega^{+}_{\underline{\mathbf{h}}(\kappa)}$ and $\Omega^{-}_{\underline{\mathbf{h}}(\kappa)}$ are canonical periods in the sense of* [Vat99, §1.3],
- *$C^{+}_{\underline{\mathbf{h}}(\kappa)}$ and $C^{-}_{\underline{\mathbf{h}}(\kappa)}$ are non-zero constants that only depend on κ and $C^{+}_{\underline{\mathbf{h}}(k_0)} = C^{-}_{\underline{\mathbf{h}}(k_0)} = 1$,*
- *the sign $\pm$ is determined so as to ensure that $(-1)^{(j-1)}\eta(-1) = \pm 1$.*

Proof. See [Bel12, Theorem 3 and (4)]. □

Remark 4.2.2. If the slope of $\underline{\mathbf{h}}$ is smaller than h, then $\mathscr{D}_{\underline{\mathbf{h}}_{/K}} \in \mathscr{H}_h \widehat{\otimes} \mathscr{A}$.

Definition 4.2.3. *For the Coleman family $\underline{\mathbf{f}}$ we have fixed above, mimicking Kobayashi* [Kob13, Kob12]*, we set*

$$\mathscr{D}^{\mathrm{naive}}_{\underline{\mathbf{f}}_{/K}} := \mathscr{D}_{\underline{\mathbf{f}}_{/\mathbb{Q}}} \cdot \mathscr{D}_{\underline{\mathbf{f}}^K_{/\mathbb{Q}}} \in \mathscr{H}_{\mathscr{A}}.$$

Here, $\underline{\mathbf{f}}^K$ is the family obtained by twisting the Coleman family $\underline{\mathbf{f}}$ by the quadratic character ϵ_K. We call $\mathscr{D}_{\underline{\mathbf{f}}_{/K}}$ the naïve base change p-adic L-function.

Remark 4.2.2 tells us that we in fact have $\mathscr{D}_{\underline{\mathbf{f}}_{/K}} \in \mathscr{H}_{2k-2} \widehat{\otimes} \mathscr{A}$.

The naïve base change p-adic L-function is then characterized by the following interpolation property:

For every $\kappa \in I$, any $j \in \mathbb{Z} \cap [1, \kappa-1]$ with $j \equiv k/2 \bmod 2$, and all *even* Dirichlet characters η of conductor p^r (we allow $r = 0$),

$$(6)\quad (\psi_\kappa \otimes \eta\chi_{\rm cyc}^{j-1})(\mathscr{D}^{\rm naive}_{\underline{\mathbf{f}}_{/K}}) = \Gamma(j)^2\, \mathcal{V}(\underline{\mathbf{f}}(\kappa)^\circ, \eta, j)^2\, \tau(\eta)^2\, \frac{p^{2r(j-1)}}{\beta(\kappa)^{2r}} \times \frac{L(\underline{\mathbf{f}}(\kappa)^\circ, \eta^{-1}, j)}{(2\pi i)^{2j}} \times \frac{C^\varepsilon_{\underline{\mathbf{f}}(\kappa)} C^\varepsilon_{\underline{\mathbf{f}}(\kappa)^K}}{\Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)} \Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)^K}}$$

where $\varepsilon \in \{\pm\}$ is the sign of $(-1)^{k/2-1}$.

4.3. **A factorization formula.** We will be working with the following locally analytic functions on $U \times (1 + p\mathbb{Z}_p)$:

Definition 4.3.1. *Given a locally analytic distribution $\mathscr{D}$ on $\Gamma_{\rm cyc}$, we set*

$$L_p(\mathscr{D}, s) := \langle \chi_{\rm cyc}\rangle^{s-1}\omega^{k/2-1}\,(\mathscr{D})$$

where ω is the Teichmüller character. We define $L_p^{\rm RS}(\underline{\mathbf{f}}, \kappa, s)$ and $L_p^{\rm naive}(\underline{\mathbf{f}}_{/K}, \kappa, s)$ on $U \times (1 + p\mathbb{Z}_p)$ by setting

$$L_p^{\rm RS}(\underline{\mathbf{f}}_{/K}, \kappa, s) := \beta^{-1}\lambda_N(\underline{\mathbf{f}})^{-1} i^{\kappa-1} N|D_K|^{-1/2}(-1)^{k/2-1} L_p(\mathscr{D}^{\rm RS}_{\underline{\mathbf{f}}_{/K}}, s)\Big|_{w=(1+p)^{\kappa-k}-1},$$

$$L_p^{\rm naive}(\underline{\mathbf{f}}_{/K}, \kappa, s) := L_p(\mathscr{D}^{\rm naive}_{\underline{\mathbf{f}}_{/K}}, s)\Big|_{w=(1+p)^{\kappa-k}-1}.$$

We also set

$$L^{\rm Kob}_{p,\beta(\kappa)}(\underline{\mathbf{f}}(\kappa)^\circ_{/K}, s) := \beta(\kappa)\,\lambda_N(\underline{\mathbf{f}}(\kappa)^\circ)\,\mathcal{E}(\underline{\mathbf{f}}(\kappa))\,\mathcal{E}^*(\underline{\mathbf{f}}(\kappa))\,L_p^{\rm RS}(\underline{\mathbf{f}}_{/K}, \kappa, s)$$

for each choice of $\kappa \in I$. In the particular case when $\kappa = k$, we shall write $L^{\rm Kob}_{p,\beta}(f_{/K}, s)$ in place of $L^{\rm Kob}_{p,\beta(k)}(\underline{\mathbf{f}}(k)^\circ_{/K}, s)$.

See Remark 4.3.7 below for a comparison of $L^{\rm Kob}_{p,\beta(\kappa)}(\underline{\mathbf{f}}(\kappa)^\circ_{/K}, s)$ to Nekovář's p-adic L-function in the p-ordinary set up.

Remark 4.3.2. Observe that the p-adic multipliers $\mathcal{E}(\underline{\mathbf{f}}(\kappa))\,\mathcal{E}^*(\underline{\mathbf{f}}(\kappa))$ do not vary continuously, the p-adic L-functions $L^{\rm Kob}_{p,\beta(\kappa)}(\underline{\mathbf{f}}(\kappa)^\circ_{/K}, s)$ do not interpolate as $\kappa \in I$ varies.

Let us consider the meromorphic function

$$R_{\underline{\mathbf{f}}/K} := L_p^{\rm RS}(\underline{\mathbf{f}}_{/K}, \kappa, s)/L_p^{\rm naive}(\underline{\mathbf{f}}_{/K}, \kappa, s).$$

Notice that for any $\kappa_0 \in I$, the specialization $L_p(\underline{\mathbf{f}}_{/K}, \kappa_0, s)$ is non-zero, so $R_{\underline{\mathbf{f}}/K}(\kappa_0, s)$ is a meromorphic function in s.

Lemma 4.3.3. *The meromorphic function $r(\kappa) := R_{\underline{\mathbf{f}}/K}(\kappa, k/2)$ (in the variable κ) specializes to*

$$\frac{\langle N\rangle^{k-\kappa}\langle 2\rangle^{-\kappa+k}2^{1-k}(-1)^{k/2}|D_K|^{1/2}}{\beta(\kappa)\lambda_N(\underline{\mathbf{f}}(\kappa))\mathcal{E}(\underline{\mathbf{f}}(\kappa))\,\mathcal{E}^*(\underline{\mathbf{f}}(\kappa))\,\langle \underline{\mathbf{f}}(\kappa)^\circ, \underline{\mathbf{f}}(\kappa)^\circ\rangle_N} \times \frac{\Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)}\,\Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)^K}}{C^\varepsilon_{\underline{\mathbf{f}}(\kappa)}\,C^\varepsilon_{\underline{\mathbf{f}}(\kappa)}}$$

$$(7)\qquad = \frac{\langle N\rangle^{k-\kappa}(-1)^{k/2}|D_K|^{1/2}}{\langle 2\rangle^{\kappa-k}2^{k-1}\langle \underline{\mathbf{f}}(\kappa), \underline{\mathbf{f}}(\kappa)\rangle_{N,p}} \times \frac{\Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)}\,\Omega^\varepsilon_{\underline{\mathbf{f}}(\kappa)^K}}{C^\varepsilon_{\underline{\mathbf{f}}(\kappa)}\,C^\varepsilon_{\underline{\mathbf{f}}(\kappa)^K}}$$

whenever $\kappa \in I$. Here, $\langle\, ,\, \rangle_{N,p}$ is the Petersson inner product at level $\Gamma_0(Np)$.

Proof. This is immediate on comparing the interpolation formulae for $L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K},\kappa,s)$ and $L_p^{\mathrm{naive}}(\underline{\mathbf{f}}_{/K},\kappa,s)$ at $s=k/2$ and $\kappa\in I$. The equality in (7) follows from the following well-known comparison of Petersson inner products:

$$\langle\underline{\mathbf{f}}(\kappa),\underline{\mathbf{f}}(\kappa)\rangle_{N,p}=\mathbb{B}(\kappa)\lambda_N(\underline{\mathbf{f}}(\kappa))\mathcal{E}(\underline{\mathbf{f}}(\kappa))\,\mathcal{E}^*(\underline{\mathbf{f}}(\kappa))\,\langle\underline{\mathbf{f}}(\kappa)^\circ,\underline{\mathbf{f}}(\kappa)^\circ\rangle_N\,.$$

□

Lemma 4.3.4. $R_{\underline{\mathbf{f}}/K}=\langle N\rangle^{2s-k}r(\kappa)$.

Proof. The interpolation formulae for $L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K},\kappa,s)$ and $L_p^{\mathrm{naive}}(\underline{\mathbf{f}}_{/K},\kappa,s)$ (given by taking $\eta=\mathbb{1}$ and $j\equiv k/2 \bmod (p-1)$ in Theorems 4.2.1 and 4.1.4, so that the character $\eta\langle\chi_{\mathrm{cyc}}\rangle^{j-1}\omega^{k/2-1}$ is crystalline at p) together with Lemma 4.3.3 show that

$$L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K},\kappa,j)=\langle N\rangle^{2j-\kappa}r(\kappa)L_p^{\mathrm{naive}}(\underline{\mathbf{f}}_{/K},\kappa,j)$$

for every $\kappa\in I$ and $j\in\mathbb{Z}\cap[1,\kappa-1]$ with $j\equiv k/2 \bmod (p-1)$. The asserted equality follows from the density of these specializations. □

Corollary 4.3.5. $R_{\underline{\mathbf{f}}/K}(k,s)=\dfrac{\langle N\rangle^{2s-k}(-1)^{k/2}|D_K|^{1/2}}{2\left\langle f^\beta,f^\beta\right\rangle_{N,p}}\Omega_f^\varepsilon\,\Omega_{f^K}^\varepsilon$. *In particular,*

$$R_{\underline{\mathbf{f}}/K}(k,k/2)=(-1)^{k/2}\,\frac{\Omega_f^\varepsilon\,\Omega_{f^K}^\varepsilon|D_K|^{1/2}}{2\left\langle f^\beta,f^\beta\right\rangle_{N,p}}\,.$$

Corollary 4.3.6.

$$L_{p,\beta}^{\mathrm{Kob}}(f_{/K},s)=\frac{\langle N\rangle^{2s-k}(-1)^{k/2}|D_K|^{1/2}}{2\left\langle f,f\right\rangle_N}\,\Omega_f^\varepsilon\,\Omega_{f^K}^\varepsilon\,L_p(f_{/\mathbb{Q}}^\beta,s)L_p(f_{/\mathbb{Q}}^{\beta,K},s)\,.$$

Proof. This is an immediate consequence of Lemma 4.3.3 and 4.3.4, on recalling that our choices enforce the requirement that $C_{f^\beta}^\pm=C_{f^\beta,K}^\pm=1$. □

Remark 4.3.7. Only in this remark, $\underline{\mathbf{h}}$ denotes a primitive Hida family of tame level N and U_p-eigenvalue $\mathbb{a}$. We let h denote its specialization to weight $2r$; suppose h is old at p and let us write α for the U_p-eigenvalue on h. In this situation, Nekovář in [Nek95, I.5.10] constructed a two-variable p-adic L-function associated to h. We let $L_p^{\mathrm{Nek}}(h_{/K},s)$ denote its restriction to cyclotomic characters.

In this particular case, the distribution $\mathscr{D}_{\underline{\mathbf{h}}}^{\mathrm{RS}}$ was constructed by Hida and it enjoys an interpolation property that is identical to one recorded in Theorem 4.1.4. One may specialize $L_p^{\mathrm{RS}}(\underline{\mathbf{h}}_{/K},\kappa,s)$ to the p-stabilized form h and obtain a p-adic L-function $L_{p,\alpha}^{\mathrm{Kob}}(h^\circ{}_{/K},s)$ as above. One may compare the interpolation formulae for the respective distributions giving rise to $L_p^{\mathrm{Nek}}(h_{/K},s)$ and $L_{p,\alpha}^{\mathrm{Kob}}(h^\circ{}_{/K},s)$ to deduce that

$$L_{p,\alpha}^{\mathrm{Kob}}(h^\circ{}_{/K},s+r-1)=L_p^{\mathrm{Nek}}(h_{/K},s)\,.$$

5. Proofs of Theorem 1.1.1 and Corollary 1.1.2

We shall assume until the end of this article that $K\neq\mathbb{Q}(i),\mathbb{Q}(\sqrt{-3})$. Notice in particular that $u_K=1$.

5.1. **p-adic Gross–Zagier formula for non-ordinary eigenforms at non-critical slope.** Suppose $g = \sum_{n=1} a_n(g)q^n \in S_{2r}(\Gamma_0(N))$ is a normalized eigenform. We let $a, b \in \overline{\mathbb{Q}}$ denote the roots of its Hecke polynomial $X^2 - a_p(g)X + p^{2r-1}$ at p. Suppose that $v_p(\iota_p(a_p(g))) > 0$ and assume that

$$0 < h := v_p(\iota_p(b)) < v_p(\iota_p(a))$$

so that we have $2h < 2r - 1$. Let $g^b \in S_{2r}(\Gamma_0(Np))$ denote the p-stabilization corresponding to the Hecke root b and let $\underline{\mathbf{g}}$ be a Coleman family which admits g^b as its specialization in weight $2r$. Theorem 4.1.4 applies and equips us with a two-variable p-adic L-function $L_p^{\mathrm{RS}}(\underline{\mathbf{g}}_{/K}, \kappa, s)$. Let us set

$$L_{p,b}^{\mathrm{Kob}}(g_{/K}, s) := b\,\lambda_N(g)\,\mathcal{E}(g)\,\mathcal{E}^*(g)\,L_p^{\mathrm{RS}}(\underline{\mathbf{g}}_{/K}, 2r, s)$$

as in Definition 4.3.1. The following p-adic Gross–Zagier formula is Kobayashi's work [Kob19] in progress.

Theorem 5.1.1 (Kobayashi)**.**

$$\frac{d}{ds} L_{p,b}^{\mathrm{Kob}}(g_{/K}, s)\big|_{s=r} = \left(1 - \frac{p^{\frac{w}{2}-1}}{b}\right)^4 \frac{h_{b,K}^{\mathrm{Nek}}(z_g, z_g)}{(4|D_K|)^{\frac{w}{2}-1}}\,.$$

The corollary below is a restatement of Theorem 5.1.1, taking the diagram (4) and Theorem 3.5.4 into account. Recall that we have to assume $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ since we rely on Kobayashi's results here, so that $u_K = 1$.

Corollary 5.1.2. *For each $\kappa \in I$ as in Section 3.4 with $\kappa \geq 2k$,*

$$\frac{d}{ds} L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K}, \kappa, s)\big|_{s=\frac{\kappa}{2}} = \frac{\left(1 - \frac{p^{\frac{\kappa}{2}-1}}{b}\right)^4}{(4|D_K|)^{\frac{\kappa}{2}-1}} \cdot \mathfrak{h}_{\underline{\mathbf{f}}(\kappa)^\circ, \beta(\kappa), K}(z_{\underline{\mathbf{f}}(\kappa)^\circ}, z_{\underline{\mathbf{f}}(\kappa)^\circ})$$

$$= \psi_\kappa \circ \mathfrak{h}_{\underline{\mathbf{f}},K}\left(\mathscr{Z}_{\underline{\mathbf{f}}}, \mathscr{Z}_{\underline{\mathbf{f}}}\right).$$

Remark 5.1.3. Note that the assumption that $\kappa \geq 2k$ guarantees that we have $2v_p(\beta(\kappa)) < \kappa - 1$, as required to apply Kobayashi's Theorem 5.1.1.

The reason why we record this trivial alteration of Theorem 5.1.1 here is because both sides of the asserted equality interpolate well as κ varies (unlike its predecessor Theorem 5.1.1). See Remark 4.3.2.

5.2. **$\mathscr{A}$-adic Gross–Zagier formula.** Recall the Coleman family $\underline{\mathbf{f}}$ over the affinoid algebra $\mathscr{A} = \mathscr{A}(U)$ from Section 2.2. Recall also the $\mathscr{A}$-valued cyclotomic height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$ we have introduced in Section 3.3 and the universal Heegner point $\mathscr{Z}_{\underline{\mathbf{f}}} \in (V_{\underline{\mathbf{f}}_{/K}}, \mathbb{D}_\beta)$ given as in Theorem 3.5.4. Recall finally also the two-variable p-adic L-function $L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K}, \kappa, s)$ from Section 4.

Definition 5.2.1. *Let us write $H_{\underline{\mathbf{f}},K}$ for the Amice transform of the height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$. In explicit terms,*

$$H_{\underline{\mathbf{f}},K}(x, y) := \mathfrak{h}_{\underline{\mathbf{f}},K}(x, y)\big|_{w=(1+p)^{\kappa-k}-1}\,.$$

Theorem 5.2.2 ($\mathscr{A}$-adic Gross–Zagier formula)**.** *With the notation as above, the following identity is valid in $\mathscr{A}$:*

$$\frac{\partial}{\partial s} L_p^{\mathrm{RS}}\left(\underline{\mathbf{f}}_{/K}, \kappa, s+\frac{\kappa-k}{2}\right)\Big|_{s=\frac{k}{2}} = H_{\underline{\mathbf{f}},K}(\mathscr{Z}_{\underline{\mathbf{f}}}, \mathscr{Z}_{\underline{\mathbf{f}}})\,.$$

Proof. Consider the difference

$$\mathfrak{D}(\kappa) := \frac{\partial}{\partial s} L_p^{\mathrm{RS}}\left(\underline{\mathbf{f}}_{/K}, \kappa, s+\frac{\kappa-k}{2}\right)\Big|_{s=\frac{k}{2}} - H_{\underline{\mathbf{f}},K}(\mathscr{Z}_{\underline{\mathbf{f}}}, \mathscr{Z}_{\underline{\mathbf{f}}})\,.$$

It follows from the interpolative properties of $L_p^{\mathrm{RS}}(\underline{\mathbf{f}}_{/K}, \kappa, s)$, the $\mathscr{A}$-adic height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$ outlined in (4) and that of the universal Heegner cycle (Theorem 3.5.4) together with Corollary 5.1.2 show that

$$\mathfrak{D}(\kappa) = 0,\ \forall\ \kappa \in I \cap \mathbb{Z}_{\geq 2k}\,.$$

By the density of $I \cap \mathbb{Z}_{\geq 2k}$ in the affinoid U, we conclude that $\mathfrak{D}$ is identically zero, as required. □

5.3. **Proof of Theorem 1.1.1.** On specializing the statement of Theorem 5.2.2 ($\mathscr{A}$-adic Gross–Zagier formula) to $\kappa = k$ and relying once again on the interpolative properties of the $\mathscr{A}$-adic height pairing $\mathfrak{h}_{\underline{\mathbf{f}},K}$ and that of the universal Heegner cycle $\mathscr{Z}_{\underline{\mathbf{f}}}$, the proof of Theorem 1.1.1 follows at once. □

5.4. **Proof of Corollary 1.1.2.** Recall that we are assuming that the weight $k = 2$. Recall also that $A_f/\mathbb{Q}$ stands for the abelian variety of GL_2-type that the Eichler-Shimura congruences associate to f and that we assume that $L(f/\mathbb{Q}, s)$ has a simple zero at $s = 1$.

It follows from the classical (complex) Gross–Zagier formula and Theorem 1.1.1 that

$$\frac{d}{ds} L_{p,\beta}^{\mathrm{Kob}}(f_{/K}, s)\big|_{s=1} = \left(1-\frac{1}{\beta}\right)^4 \frac{L'(A_f/K,1)\,|D_K|^{1/2}}{\langle P_f, P_f\rangle_{\infty,K}\, 8\pi^2\, \langle f,f\rangle_N} h_{\beta,K}^{\mathrm{Nek}}(P_f, P_f) \tag{8}$$

where we recall that $P_f \in A_f(K)$ is the Heegner point and $\langle\, ,\, \rangle_{\infty,K}$ is the Néron-Tate height pairing over K. Since we know in our set up that $\mathrm{Tr}_{K/\mathbb{Q}} P_f$ is non-torsion, it is a non-zero multiple of P within the one-dimensional $\mathbb{Q}$-vector space $A_f(\mathbb{Q}) \otimes \mathbb{Q}$. We may therefore replace in (8) the height pairings of P_f over K with those of P over $\mathbb{Q}$ to deduce that

$$\frac{d}{ds} L_{p,\beta}^{\mathrm{Kob}}(f_{/K}, s)\big|_{s=1} = \left(1-\frac{1}{\beta}\right)^4 \frac{L'(A_f/K,1)|D_K|^{1/2}}{\langle P, P\rangle_{\infty}\, 8\pi^2\, \langle f,f\rangle_N} h_{\beta,\mathbb{Q}}^{\mathrm{Nek}}(P, P) \tag{9}$$

On the other hand we have

$$L'(A_f/K,1) = L'(A_f/\mathbb{Q},1) L(A_f^K/\mathbb{Q},1) \tag{10}$$

and

$$\frac{d}{ds} L_{p,\beta}^{\mathrm{Kob}}(f_{/K}, s)\big|_{s=1} = \frac{-4\pi^2 \Omega_f^+ \Omega_{f^K}^+ |D_K|^{1/2}}{8\pi^2 \langle f,f\rangle_N} \times \frac{d}{ds} L_{p,\beta}(f_{/\mathbb{Q}}, s)\big|_{s=1} \tag{11}$$

$$\times \left(1-\frac{1}{\beta}\right)^2 \frac{L(A_f^K/\mathbb{Q},1)}{-2\pi i\, \Omega_{f^K}^+} \tag{12}$$

by the definition of $L_{p,\beta}^{\mathrm{Kob}}(f_{/K},s)$ and the interpolation property of $L_{p,\beta}(f_{/\mathbb{Q}}^{K},s)$. Plugging the identities (10) and (11) in (9), the desired equality follows. $\square$

6. Applications

We shall illustrate various applications of the p-adic Gross–Zagier formula at critical slope (Theorem 1.1.1 and Corollary 1.1.2). These were already recorded in Section 1 as Theorems 1.1.5, 1.1.6 and 1.1.8. Before we give proofs of these results, we set some notation and record a number of preliminary results.

6.1. **Perrin-Riou's big logarithm map and p-adic L-functions.** Until the end of this article, we assume that $f \in S_2(\Gamma_0(N))$ is a newform that does not admit a θ-critical p-stabilization. Recall that $A_f/\mathbb{Q}$ denotes the abelian variety of GL_2-type that the Eichler-Shimura congruences associate to f. Our assumption that $L(f/\mathbb{Q},s)$ has a simple zero at $s=1$ is still in effect. We assume also that the residual representation $\overline{\rho}_f$ (associated to the $\mathfrak{P}$-adic representation attached to f) is absolutely irreducible.

Let $\mathfrak{P}$ denote the prime of K_f above that is induced by the embedding ι_p and set $T := (\varprojlim A_f(\overline{\mathbb{Q}})[\mathfrak{P}^n]) \otimes_{\mathcal{O}_{f,\mathfrak{P}}} \mathcal{O}$ and $V = T \otimes_{\mathcal{O}} E$. Here, we recall that E is a finite extension of $K_{f,\mathfrak{P}}$ that contains both α and β (where α is the root of $X^2 - a_p(f)X + p$ which is a p-adic unit and $\beta = p/\alpha$ is the other root) and $\mathcal{O}$ is its ring of integers. Since we assumed $p>2$, the Fontaine-Laffaille condition holds true for V. In particular, there is an integral Dieudonné module $\mathbb{D}_{\mathrm{cris}}(T) \subset \mathbb{D}_{\mathrm{cris}}(V)$ constructed as in [Ber04, §IV]. We fix a φ-eigenbasis $\{\omega_\alpha,\omega_\beta\}$ of $\mathbb{D}_{\mathrm{cris}}(T)$. Let

$$\mathfrak{Log}_V : H^1(\mathbb{Q}_p, V \otimes \Lambda^\iota) \otimes \mathscr{H} \longrightarrow \mathbb{D}_{\mathrm{cris}}(V) \otimes \mathscr{H}$$

denote Perrin-Riou's big dual exponential map. We let

$$\mathfrak{L}_{\mathrm{PR}} := \mathfrak{Log}_V \circ \mathrm{res}_p\,(\mathbb{BK}_1) \in \mathbb{D}_{\mathrm{cris}}(V) \otimes \mathscr{H}$$

denote Perrin-Riou's vector valued p-adic L-function, where $\mathbb{BK}_1 \in H^1(\mathbb{Q}, V \otimes \Lambda^\iota)$ is the Beilinson-Kato element. Set $\mathscr{H}_E := \mathscr{H} \otimes_{\mathbb{Q}_p} E$ and define $\mathfrak{L}_{\mathrm{PR}}^{(\alpha)}, \mathfrak{L}_{\mathrm{PR}}^{(\beta)} \in \mathscr{H}_E$ as the coordinates of $\mathfrak{L}_{\mathrm{PR}}$ with respect to the basis $\{\omega_\alpha,\omega_\beta\}$, so that we have

$$\mathfrak{L}_{\mathrm{PR}} = \mathfrak{L}_{\mathrm{PR}}^{(\alpha)}\,\omega_\alpha + \mathfrak{L}_{\mathrm{PR}}^{(\beta)}\,\omega_\beta\,.$$

Note that $\mathfrak{L}_{\mathrm{PR}}^{(\alpha)}$ and $\mathfrak{L}_{\mathrm{PR}}^{(\beta)}$ are well-defined only up to multiplication by an element of $\mathcal{O}^\times$.

Definition 6.1.1. *Set $\mathscr{D}_{f^\beta_{/\mathbb{Q}}} := \psi_2(\mathscr{D}_{\mathbf{f}_{/\mathbb{Q}}}) \in \mathscr{D}_{k-1}(\Gamma) \otimes E$. Associated to the other (p-ordinary) stabilization f^α of f, we also have the Mazur–Swinnerton-Dyer measure $\mathscr{D}_{f^\alpha_{/\mathbb{Q}}} \in \mathscr{D}_0(\Gamma) \otimes E$. We remark that the measure $\mathscr{D}_{f^\alpha_{/\mathbb{Q}}}$ is characterized by an interpolation formula identical to that for $\mathscr{D}_{f^\beta_{/\mathbb{Q}}}$ (which does* not *characterize $\mathscr{D}_{f^\beta_{/\mathbb{Q}}}$ itself), exchanging every α in the formula with β and vice versa.*

For $\lambda = \alpha, \beta$, we set $L_{p,\lambda}(f,s) := L_p(\mathscr{D}_{f^\lambda_{/\mathbb{Q}}}, s)$.

The following result is due to Kato (when $\lambda = \alpha$) and has been announced by Hansen (when $\lambda = \beta$). See also related work by Ochiai [Och18].

Theorem 6.1.2. *Suppose that f^β is non-critical. Then for each $\lambda \in \{\alpha, \beta\}$, there exists $c_\lambda \in \mathcal{O}^\times$ with $\mathfrak{L}_{\mathrm{PR}}^{(\lambda)} = c_\lambda \cdot \mathscr{D}_{f^\lambda_{/\mathbb{Q}}}$.*

Remark 6.1.3. When $K_f = \mathbb{Q}$ and A_f is an elliptic curve, there is a choice for the φ-eigenbasis of $\mathbb{D}_{\mathrm{cris}}(V)$ with which we can take $c_\lambda = 1$ for $\lambda = \alpha, \beta$. In this remark, we explain how to make this choice.

Let $\mathscr{A}_{/\mathbb{Z}}$ be a minimal Weierstrass model of the elliptic curve A_f. Let $\omega_{\mathscr{A}}$ denote a Néron differential that is normalized as in [PR95, §3.4] and is such that $\Omega^+_{A_f} := \int_{A_f(\mathbb{R})} \omega_{\mathscr{A}} > 0$. Let $\omega_{\mathrm{cris}} \in \mathbb{D}_{\mathrm{cris}}(V)$ denote the element that corresponds to $\omega_{\mathscr{A}}$ under the comparison isomorphism. The eigenbasis $\{\omega_\alpha, \omega_\beta\}$ is then given by the requirement that

$$\omega_\alpha + \omega_\beta = \omega_{\mathrm{cris}} .$$

6.1.1. *Proof of Theorem 1.1.6 (non-triviality of p-adic heights).* Suppose on the contrary that both $h^{\mathrm{Nek}}_{\alpha,\mathbb{Q}}$ and $h^{\mathrm{Nek}}_{\beta,\mathbb{Q}}$ were trivial. It follows from Corollary 1.1.2 and Perrin-Riou's p-adic Gross–Zagier formula for the slope-zero p-adic L-function $L_p(\mathscr{D}_{f^\alpha/\mathbb{Q}}, s)$ that

$$\mathbb{1}\,(\mathfrak{L}'_{\mathrm{PR}}) = 0 .$$

Using [PR93, Proposition 2.2.2], we conclude that $\log_V(\mathrm{BK}_1) = 0$, or equivalently, that $\mathrm{res}_p(\mathrm{BK}_1) = 0$. Since $\mathrm{ord}_{s=1} L(f/\mathbb{Q}, s) = 1$, the theorem of Kolyvagin-Logachev shows that the compositum of the arrows

$$A_f(\mathbb{Q}) \otimes_{\mathcal{O}_f, \iota_p} E \xrightarrow{\sim} H^1_{\mathrm{f}}(\mathbb{Q}, V) \xrightarrow{\mathrm{res}_p} H_{\mathrm{f}}(\mathbb{Q}_p, V) = A_f(\mathbb{Q}_p) \otimes_{\mathcal{O}_{f,\mathfrak{P}}} E$$

is injective. It follows that $\mathrm{BK}_1 \in H^1_{\mathrm{f}}(\mathbb{Q}, V)$ is a torsion class, contradicting [Büy17, Theorem 1.2][9]. $\square$

6.2. **Birch and Swinnerton-Dyer formula for analytic rank one (Proof of Theorem 1.1.8).** Recall the set $\Sigma := \{\sigma : K_f \hookrightarrow \overline{\mathbb{Q}}\}$ of embeddings of K_f into $\overline{\mathbb{Q}}$. Each embedding σ extends to $\sigma : K_f(\alpha) \hookrightarrow \overline{\mathbb{Q}}$; fix one such extension. Recall that $\mathfrak{P}$ is the prime induced by the embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, which we extend to an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$. To save ink, let us set λ^σ in place of $\iota_p \circ \sigma(\lambda)$, where $\lambda \in \{\alpha, \beta\}$.

For each $\sigma \in \Sigma$, the field $\sigma(K_f)$ is the Hecke field K_{f^σ} of f^σ and let $\mathfrak{P}_\sigma \subset \sigma(K_f)$ denote its prime induced by ι_p. Let $E = K_{f,\mathfrak{P}}(\iota_p(\alpha))$ denote the extension of $K_{f,\mathfrak{P}}$ generated by $\iota_p(\alpha)$ and let $\mathcal{O}$ denote its ring of integers, $\mathfrak{m}$ its maximal ideal. We shall set $E^\sigma := K_{f^\sigma, \mathfrak{P}_\sigma}(\alpha^\sigma)$ to ease notation and write $\mathcal{O}^\sigma$ for its ring of integers, $\mathfrak{m}^\sigma$ for its maximal ideal. Let us write $T^\sigma := \varprojlim A_f(\overline{\mathbb{Q}})[\mathfrak{P}_\sigma^n]$, where the action of $\mathfrak{P}_\sigma$ act on A_f is induced from $\sigma(K_f) \xrightarrow{\sigma^{-1}} K_f$, and we set $V^\sigma = T^\sigma \otimes E^\sigma$.

We retain the set up in the previous section, except that we write for each $\sigma \in \Sigma$

$$\mathfrak{L}^\sigma_{\mathrm{PR}} \in \mathbb{D}_{\mathrm{cris}}(V^\sigma) \otimes \mathscr{H}_{E^\sigma}$$

for Perrin-Riou's vector valued p-adic L-function associated to f^σ and the prime $\mathfrak{P}_\sigma$ of K_f.

[9] The proof of this result is provided in op. cit. only when $K_f = \mathbb{Q}$, but the argument carries over to treat the general case.

Proposition 6.2.1. *Suppose the Iwasawa main conjecture holds true for each $f^\sigma_{/\mathbb{Q}}$. Then for each $\sigma \in \Sigma$, there exists $\lambda^\sigma \in \{\alpha^\sigma, \beta^\sigma\}$ such that p-adic height pairing*

$$h^{\mathrm{Nek}}_{\lambda^\sigma,\mathbb{Q}} : H^1_{\mathrm{f}}(\mathbb{Q}, V^\sigma) \otimes H^1_{\mathrm{f}}(\mathbb{Q}, V^\sigma) \longrightarrow E^\sigma$$

is non-trivial.

Proof. If $\iota_p \circ \sigma \circ \iota_\infty^{-1}(a_p)$ is a p-adic unit, then this assertion is already proved in Theorem 1.1.6. Otherwise, the assertion follows (still as in the proof of Theorem 1.1.6) from Kobayashi's p-adic Gross-Zagier formulae [Kob12, Theorem 3], validity of main conjectures up to μ-invariants in that set up (which we assume) and [Büy17, Theorem 1.2]. $\square$

Proof of Theorem 1.1.8. Since we assume the validity of main conjectures for f^σ, Perrin-Riou's leading term formulae[10] for her module of p-adic L-functions[11] in [PR93, §3] together with Theorem 6.1.2 and Proposition 6.2.1 show that the $\mathfrak{m}^\sigma$-adic Birch and Swinnerton-Dyer conjecture (which corresponds to the statement $\mathrm{BSD}_{\mathbb{D}_\lambda}(V)$ in [PR93, Proposition 3.4.6]) for A_f is true up to $\mathfrak{m}^\sigma$-adic units:

(13)
$$\mathrm{ord}_{\mathfrak{m}^\sigma}\left((1-1/\lambda^\sigma)^{-2}\,\frac{L'_{p,\lambda^\sigma}(f^\sigma/\mathbb{Q},1)}{\mathrm{Reg}_{\mathfrak{P}_\sigma}(A_f/\mathbb{Q})}\right) = \mathrm{length}_{\mathcal{O}^\sigma}\left(\mathrm{III}(A_f/\mathbb{Q})[\mathfrak{P}_\sigma^\infty]\right) + \mathrm{ord}_{\mathfrak{m}^\sigma}\mathrm{Tam}(A_f/\mathbb{Q})$$

for every $\sigma \in \Sigma$. Here,

$$\mathrm{Reg}_{\mathfrak{P}_\sigma}(A_f/\mathbb{Q}) = \frac{h^{\mathrm{Nek}}_{\lambda^\sigma,\mathbb{Q}}(P_{f^\sigma}, P_{f^\sigma})}{[A_f(\mathbb{Q}) \otimes_{\mathcal{O}_f} \mathcal{O}^\sigma : \mathcal{O}^\sigma \cdot P_{f^\sigma}]}\,. \tag{14}$$

and notice that the terms concerning the torsion groups $A_f(\mathbb{Q})[\mathfrak{P}_\sigma^\infty]$ and $A_f^\vee(\mathbb{Q})[\mathfrak{P}_\sigma^\infty]$ are omitted from this formula, as they are both trivial since we assume that $\overline{\rho}_f = A_f(\overline{\mathbb{Q}})[\mathfrak{P}_\sigma]$ is absolutely irreducible.

Applying either Corollary 1.1.2 (if the p-adic valuation of λ^σ is 1), or Perrin-Riou's p-adic Gross–Zagier formula at slope-zero (if λ^σ is a p-adic unit) or Kobayashi's p-adic Gross-Zagier formula at supersingular primes (if the p-adic valuation of λ^σ is positive but less than 1), we see that

$$-\frac{L'(f^\sigma/\mathbb{Q},1)}{\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})\,2\pi i\,\Omega^+_{f^\sigma}} = (1-1/\lambda^\sigma)^{-2}\,\frac{L'_{p,\lambda^\sigma}(f^\sigma,1)}{\mathrm{Reg}_{\mathfrak{P}_\sigma}(A_f/\mathbb{Q})} \tag{15}$$

where $\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q}) := \langle P_{f^\sigma}, P_{f^\sigma}\rangle_\infty/[A_f(\mathbb{Q}) \otimes_{\mathcal{O}_f} \mathcal{O}^\sigma : \mathcal{O}^\sigma \cdot P_{f^\sigma}]$. We remark that this equality takes place in the field $E^\sigma = \sigma(K_f)_{\mathfrak{P}_\sigma}(\alpha^\sigma)$. Combining (13) and (15), we infer that

$$\mathrm{ord}_{\mathfrak{m}^\sigma}\left(-\frac{L'(f^\sigma/\mathbb{Q},1)}{\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})2\pi i\,\Omega^+_{f^\sigma}}\right) = \mathrm{length}_{\mathcal{O}^\sigma}\left(\mathrm{III}(A_f/\mathbb{Q})[\mathfrak{P}_\sigma^\infty]\right) + \mathrm{ord}_{\mathfrak{m}^\sigma}\mathrm{Tam}(A_f/\mathbb{Q})\,.$$

[10] Perrin-Riou's Proposition 3.4.6 in [PR93] is written for the p-adic Tate module of an elliptic curve, but it works verbatim for the Galois representation T^σ (the $\mathfrak{P}_\sigma$-adic Tate-module of A_f).

[11] Note that the element $\mathfrak{L}^\sigma_{\mathrm{PR}}$ we have introduced above is a generator of this module since we assume the truth of main conjectures.

The proof of the first assertion in Theorem 1.1.8 follows.

We now explain the proof of its second portion; that the Birch and Swinnerton-Dyer formula for an elliptic curve $A/\mathbb{Q}$ (satisfying the conditions of the second portion of our theorem) is valid up to p-adic units. Let f denote the newform associated to A. Based on our results above in the general case, we only need to prove that the rational number $2\pi i\,\Omega_f^+/\Omega_A^+$ is a p-adic unit. This amounts to showing that the Manin constant c_A is a p-adic unit. In our setting, this follows from [Maz78, Corollary 4.1], which states that if $p \mid c_A$, then $p^2 \mid 4N$.

□

Remark 6.2.2. Recall the σ-part $\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})$ of the regulator $\mathrm{Reg}_\infty(A_f/\mathbb{Q})$ which we defined as

$$\mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q}) := \frac{\langle P_{f^\sigma}, P_{f^\sigma}\rangle_\infty}{[A_f(\mathbb{Q}) \otimes_{\mathcal{O}_f} \mathcal{O}^\sigma : \mathcal{O}^\sigma \cdot P_{f^\sigma}]},$$

where the ring $\mathcal{O}^\sigma$ is given as above. Note that the set $\{P_{f^\sigma}\}_{\sigma\in\Sigma} \subset A_f(\mathbb{Q})$ gives rise to an orthogonal basis of $A_f(\mathbb{Q}) \otimes \mathbb{Q}$ (with respect to the archimedean height pairing), so that we have the factorization

$$\mathrm{Reg}_\infty(A_f/\mathbb{Q}) = \prod_{\sigma\in\Sigma} \mathrm{Reg}_{\infty,\sigma}(A_f/\mathbb{Q})\,.$$

6.3. **Proof of Theorem 1.1.5 (Perrin-Riou's conjecture).** We assume throughout Section 1.1.5 that $f = f_A$ is an eigenform of weight 2, which is associated to the elliptic curve $A_{/\mathbb{Q}}$ that has good ordinary reduction at p and that has analytic rank one. We also assume throughout that $\overline{\rho}_A$ is absolutely irreducible.

We shall follow the argument in the proof of Theorem 2.4(iv) of [Büy17] very closely, where the analogous assertion has been verified in the case when the prime p is a prime of good supersingular reduction. Essentially, the argument in op. cit. works verbatim, on replacing all references to Kobayashi's work with references to Corollary 1.1.2, Perrin-Riou's p-adic Gross–Zagier formula at slope zero and Theorem 1.1.6. We summarize it here for the convenience of the readers.

Let us write

$$\mathfrak{Log}_V = \mathfrak{Log}_{V,\alpha} \cdot \omega_\alpha + \mathfrak{Log}_{V,\beta} \cdot \omega_\beta\,.$$

Recall that $\omega_{\mathscr{A}}^* \in \mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V)$ stands for the unique element such that $[\omega_{\mathscr{A}}, \omega_{\mathscr{A}}^*] = 1$. We define $\log_A(\mathrm{res}_p(\mathrm{BK}_1))$ according to the identity

$$\log_V(\mathrm{res}_p(\mathrm{BK}_1)) = \log_A(\mathrm{res}_p(\mathrm{BK}_1)) \cdot \omega_{\mathscr{A}}^*\,.$$

The dual basis of $\{\omega_\alpha, \omega_\beta\}$ with respect to the pairing $[\,,\,]$ is $\{\omega_\beta^*, \omega_\alpha^*\}$, where ω_β^* (respectively, ω_α^*) is the image of $\omega_{\mathscr{A}}^*$ under the inverse of the isomorphism

$$s_{D_\beta} : D_\beta \xrightarrow{\sim} \mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{cris}}(V)$$

(respectively, under the inverse of s_{D_α}). Let $\lambda \in \{\alpha, \beta\}$ be such that the height pairing $h^{\rm Nek}_{\lambda,\mathbb{Q}}$ is non-trivial and let λ^* be given so that $\{\lambda, \lambda^*\} = \{\alpha, \beta\}$. Then,

$$\begin{aligned}(1-1/\lambda)^2 \cdot c(f) \cdot h^{\rm Nek}_{\lambda,\mathbb{Q}}(P,P) &= L'_{p,\lambda}(f,1)\\ &= \mathbb{1}\left(\mathfrak{Log}_{V,\lambda}(\partial_\lambda \mathbb{BK}_1)\right)\\ &= \left[\exp^*(\partial_\lambda {\rm BK}_1), (1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\cdot \omega^*_{\lambda^*}\right]\\ &= (1-p^{-1}\lambda^*)(1-1/\lambda^*)^{-1}\left[\exp^*(\partial_\lambda {\rm BK}_1), \omega^*_{\mathscr{A}}\right]\\ &= (1-1/\lambda)(1-1/\lambda^*)^{-1}\frac{[\exp^*(\partial_\lambda {\rm BK}_1), \log_V({\rm res}_p({\rm BK}_1))]}{\log_A({\rm res}_p({\rm BK}_1))}\\ &= -(1-1/\lambda)(1-1/\lambda^*)^{-1}\frac{h^{\rm Nek}_{\lambda,\mathbb{Q}}({\rm BK}_1,{\rm BK}_1)}{\log_A({\rm res}_p({\rm BK}_1))}.\end{aligned}$$

Here:

- The first equality follows from Perrin-Riou's p-adic Gross–Zagier formula if $\lambda = \alpha$, or else it is Corollary 1.1.2.
- The second equality follows from the definition of $\mathfrak{Log}_{V,\lambda}$ and the fact that it maps to Beilinson-Kato class to $\mathscr{D}_{f^\lambda_{/\mathbb{Q}}}$ (Theorem 6.1.2); as well as the definition of the *derived Beilinson-Kato class*
$$\partial_\lambda \mathbb{BK}_1 \in H^1_{\rm Iw}(\widetilde{\mathbb{D}}_\lambda)$$
which is given within the proof of Theorem 2.4 in [Büy17].
- The element
$$\partial_\lambda {\rm BK}_1 \in H^1_{/{\rm f}}(\mathbb{Q}_p, V) := H^1(\mathbb{Q}_p, V)/H^1_{\rm f}(\mathbb{Q}_p, V)$$
is the projection of the derived Beilinson-Kato class $\partial_\lambda \mathbb{BK}_1$ under the natural map
$${\rm pr}_0 \,:\, H^1_{\rm Iw}(\widetilde{\mathbb{D}}_\lambda) \longrightarrow H^1_{/{\rm f}}(\mathbb{Q}_p, V)$$
and the third equality follows from the explicit reciprocity laws of Perrin-Riou (as proved by Colmez) (c.f. the discussion in [BL17b, Section 2.1]).
- Fourth and fifth equalities follow from definitions (and using the fact that $\lambda\lambda^* = p$).
- The final equality follows from the Rubin-style formula proved in [BB17, Theorem 4.13] and the comparison of various p-adic heights summarized in the diagram (4).

We therefore infer that

$$\frac{h^{\rm Nek}_{\lambda,\mathbb{Q}}({\rm BK}_1,{\rm BK}_1)}{\log_A({\rm res}_p({\rm BK}_1))} = -(1-1/\alpha)(1-1/\beta)\cdot c(f)\cdot h^{\rm Nek}_{\lambda,\mathbb{Q}}(P,P)\,. \tag{16}$$

The fact that $h^{\rm Nek}_{\lambda,\mathbb{Q}}(*,*)$ and $(\log_A \circ {\rm res}_p(*))^2$ are both non-trivial quadratic forms on the one dimensional $\mathbb{Q}_p$-vector space $A(\mathbb{Q})\otimes\mathbb{Q}_p$ and combining with (16), we conclude that

$$\frac{h^{\rm Nek}_{\lambda,\mathbb{Q}}(P,P)}{\log_A({\rm res}_p(P))^2} = \frac{h^{\rm Nek}_{\lambda,\mathbb{Q}}({\rm BK}_1,{\rm BK}_1)}{\log_A({\rm res}_p({\rm BK}_1))^2} = -(1-1/\alpha)(1-1/\beta)\cdot c(f)\cdot\frac{h^{\rm Nek}_{\lambda,\mathbb{Q}}(P,P)}{\log_A({\rm res}_p({\rm BK}_1))}\,.$$

□

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