Mod 5 icosahedral representations and a conjecture of Artin

Shu Sasaki

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Introduction

Let $K$ be a number field. Let $\mathbb{A}_K$ denote the adele of $K$.

Let $\rho : G_K \to GL_n(\mathbb{C})$ be a $n$-dimensional continuous representation of the absolute Galois group $G_K$ of $K$. Let $L(\rho, s)$ be the (Artin) $L$-series

$$L(\rho, s) = \prod_v \det(1 - \rho^I_v(Frob_v)(N_v)^{-s})^{-1}$$

in $s \in \mathbb{C}$ associated to $\rho$, where by $\rho^I_v$ I mean the representation of $G_v/I_v$ on the subspace of the inertia $I_v$ invariants, $N_v = \#\mathcal{O}_K/v$, and $Frob_v$ is the arithmetic Frobenius at $v$. 
A theorem of Brauer asserts: \( L(\rho, s) \) has meromorphic continuation to the whole of \( \mathbb{C} \) (a piece of "group theory"; actually the Brauer’s theorem is the genesis of "potential automorphy" by R. Taylor).

The Artin conjecture asserts: the \( L \)-series \( L(\rho, s) \) has holomorphic continuation to \( s \in \mathbb{C} \) except for a possible pole at \( s = 1 \).

If \( \rho = \rho_1 + \rho_2 \), \( L(\rho, s) = L(\rho_1, s)L(\rho_2, s) \); so we may assume \( \rho \) is irreducible.

A conjecture of Langlands ("Langlands programme"), known more commonly as the strong Artin conjecture, predicts: there exists a cuspidal automorphic representation \( \pi \) of \( GL_n(\mathbb{A}_K) \) such that \( L(\rho, s) = L(\pi, s) \).
It is well-known that the strong Artin conjecture implies the Artin conjecture. As far as I know, D. Ramakrishnan wrote down a proof (an exercise in complex analysis).

If \( n = 1 \), this is the global class field theory: a canonical bijection between Hecke characters of \( \mathbb{A}^\times_K \) (the “automorphic side”) and Galois character (the “Galois side”) of \( G_K \).
If $n = 2$, let
\[ \text{proj} \rho : G_K \to GL_2(\mathbb{C}) \to PGL_2(\mathbb{C}) := GL_2(\mathbb{C})/\mathbb{C}^\times. \]

Then the image of $\text{proj} \rho$ is either dihedral, tetrahedral ($A_4$), octahedral ($S_4$), or icosahedral ($A_5$).

The dihedral case is due to Artin himself. The tetrahedral case, and the octahedral case with $K = \mathbb{Q}$, $\rho$ odd, is due to Langlands (“soluble base change”). Tunnell treats the octahedral case in general (still as a result of the soluble base change trick).

Except some computational evidence (Buhler, Frey, et al.), the icosahedral case was largely intractable ($A_5$ is not soluble)!
For brevity, I shall henceforth call a representation $\rho$ icosahedral if the image of $\text{proj} \rho$ is $A_5$.

If $n = 2$, $K$ is a totally real field, and $\rho$ is totally odd (i.e., the determinant of the image of complex conjugation with respect to every embedding of $K$ into $\mathbb{R}$ is -1), then the strong Artin conjecture predicts: there exists a holomorphic cusp eigenform $f$ over $K$ such that $L(f, s) = L(\rho, s)$.

What about the “even” case? Well, this amounts to finding Maass forms...
AUTOMORPHIC TWO-DIMENSIONAL GALOIS REPRESENTATIONS

Let $K$ be a totally real field. Let $f$ be a holomorphic (Hilbert) cusp eigenform over $K$ and $\pi(f)$ denote the cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$ generated by $f$.

"$f \mapsto \pi(f) = \pi \mapsto \rho_\pi$" is established by

the regular weight case: Carayol ($[K : \mathbb{Q}]$ odd, or $[K : \mathbb{Q}]$ even and $\pi$ is square-integrable at some finite place); Wiles (the ordinary case); Taylor ($[K : \mathbb{Q}]$ is even),

the parallel weight one case: Ragawski-Tunnell

the partial weight one case: Jarvis
In the following, when $K$ is totally real, I will say “a (totally odd two-dimensional) $p$-adic/mod $p$ representation $\rho$ of $G_K$ is modular”.

By this, in characteristic zero, I will mean that there exists a holomorphic cusp eigenform $f$ over $K$ such that its associated Galois representation $\rho_f : G_K \to GL_2(L)$, where $L = \mathbb{Q}_p(\{a_n(f)\})$, is isomorphic to $\rho$.

In characteristic $p$, I shall mean that the semi-simplification (i.e., the direct sum of the Jordan-Holder constituents) of the reduction

$$\bar{\rho}_f : G_K \to GL_2(\mathcal{O}_L) \twoheadrightarrow GL_2(\mathcal{O}_L/m_L) \simeq GL_2(k_L)$$

of the “model” $G_K \to GL_2(\mathcal{O}_L)$ is isomorphic to $\rho$. 
THE STRONG ARTIN CONJECTURE FOR ODD ICOSAHEDRAL REPRESENTATIONS

In 2001, Buzzard-Dickinson-Shepherd-Barron-Taylor “On icosahedral Artin representations” proved many new cases of the strong Artin conjecture for odd icosahedral $\rho : G_\mathbb{Q} \rightarrow GL_2(\mathbb{C})$.

Which was followed by Taylor “On icosahedral Artin representations II”, 2003.

These are based on Taylor’s idea to “deduce” results about weight 1 forms from results about weight 2 forms, i.e., Wiles’s idea about modularity of semi-stable elliptic curves over $\mathbb{Q}$. 
More precisely,

(0) Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$ for some $p$. Let

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_L)$$

for a finite extension $L$ of $\mathbb{Q}_p$, and let

$$\overline{\rho} : G_{\mathbb{Q}} \to GL_2(k_L)$$

be the “reduction mod $p$” of $\rho$.

(1) Prove that $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(k_L)$ is modular. This is commonly known as “Serre’s conjecture for $\overline{\rho}$.

(2) Prove that $\overline{\rho}$ modular implies $\rho$ modular. This, on the other hand, is known as Modular Lifting Theorem, or $R = T$.

(3) Combine (1) and (2) together, $\rho$ is modular.
This is how Wiles proved an semistable elliptic curve over $\mathbb{Q}$ is modular when $\rho$ is the Tate module $\rho_E : G_\mathbb{Q} \to GL(E(\overline{\mathbb{Q}}_p)[p^\infty]) \simeq GL_2(\mathbb{Z}_p)$.

(1) is given by “Langlands-Tunnell” with $p = 3$;

$$\overline{\rho}_{E,3} : G_\mathbb{Q} \to GL(E(\overline{\mathbb{Q}})[3]) \simeq GL_2(\mathbb{F}_3)$$

followed by an explicit homomorphism

$$GL_2(\mathbb{F}_3) \to GL_2(\mathbb{Z}(\sqrt{-2})) \subset GL_2(\mathbb{C})$$

is odd, irreducible, and soluble ($PGL_2(\mathbb{F}_3) \simeq S_4$). The composition is “modular” and therefore $\overline{\rho}_{E,3}$ is modular.
(2) is given by “$R = T$”; Wiles proves that, for any $p$, if $\bar{\rho}$ is a mod $p$ representation $\bar{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{F}_p)$, which is modular and whose restriction to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2p}}))$ is absolutely irreducible, then

the set $R$ of all deformations (flat at $p$) of $\bar{\rho}$ is isomorphic to

the set $T$ of all deformations of $\bar{\rho}$ arising from (in the sense of Eichler-Shimura(-Deligne)) weight 2 cusp forms.

In particular, for $p$ either 3 or 5, $\rho_E : G_\mathbb{Q} \to GL_2(\mathbb{Z}_p)$ associated to a semistable elliptic curve $E$ over $\mathbb{Q}$ gives a $\mathbb{Z}_p$-valued point of $\text{Spec} \, R$, so it gives rises to a $\mathbb{Z}_p$-valued point of $\text{Spec} \, T$, hence $\rho_E$ is modular.
Taylor’s idea (1992) (for tackling the strong Artin conjecture in the icosahedral case) was to use this trick to prove modularity of odd icosahedral $\rho : G_Q \rightarrow GL_2(\mathcal{O}_L)$.

Slightly more precisely,

prove (2) that, given a $p$-adic representation $\rho : G_Q \rightarrow GL_2(\mathcal{O}_L)$ whose *$p$-adic Hodge-Tate weights* are equal, and its $\bar{\rho} : G_Q \rightarrow GL_2(k_L)$ is modular, then $\rho$ arises from a weight one form (which is much stronger than modularity of icosahedral $\rho$ that we will need);

and prove (1) that odd icosahedral $\bar{\rho} : G_Q \rightarrow GL_2(k_L)$ is modular.
Results about modularity of mod $p$ icosahedral representations of $G_Q$.

Shepherd-Barron-Taylor (2001) If $\bar{\rho} : G_Q \to GL_2(\mathbb{F}_4)$ is unramified at 3 and 5, then $\bar{\rho}$ is modular.

The theorem is, in fact, pre-Shimura-Taniyama (Breuil-Conrad-Diamond-Taylor). After S-T, the condition at 3 can be suppressed.

If $\bar{\rho}$ is unramifed at 2 and 5, and $\bar{\rho}(\text{Frob}_2)$ has distinct eigenvalues, then $\bar{\rho}$ is modular.

Taylor (2003) If $\bar{\rho} : G_Q \to GL_2(\mathbb{F}_5)$ is “$I_3$-distinguished” and “5-distinguished”, then $\bar{\rho}$ is modular.
Of course,

Khare-Wintenberger (2009) Any odd, continuous, and irreducible $\overline{\rho}$ is modular ("Serre’s conjecture").

However, if $\text{proj } \rho$ is icosahedral, so is $\text{proj } \overline{\rho}$; and since only $\text{PSL}_2(\mathbb{F}_5)$ and $\text{PSL}_2(\mathbb{F}_4)$ are isomorphic to $A_5$, it would suffices to know modularity of $\overline{\rho}$ for $p = 2, 5$. 
Results about MLTs.

Buzzard-Taylor (1999) For any odd $p \ (p = 2$ works if combined with Dickinson’s “$R = T$ theorem”) $\rho : G_Q \to GL_2(\mathcal{O}_L)$ arises from a weight one form if $\rho : G_Q \to GL_2(\mathcal{O}_L)$ is unramified at $p$, $\rho(\text{Frob}_p)$ has distinct eigenvalues, and $\bar{\rho}$ is modular.

Buzzard (2003) For any $p$, $\rho : G_Q \to GL_2(\mathcal{O}_L)$ arises from a weight one form if $\rho$ is “potentially unramified at $p$” (i.e., $\rho(I_p)$ is finite), $\rho|_{G_p}$ is the direct sum of two characters of $G_p$ which are distinct mod $p$, and $\bar{\rho}$ is modular.

Khare (1997) $\rho : G_Q \to GL_2(\mathbb{C})$ arises from a weight one form if $\rho_p : G_Q \to GL_2(\mathbb{Q}_p) \subset GL_2(\mathbb{Q}_p) \simeq GL_2(\mathbb{C})$, when reduced mod $p$, is modular for many $p$ (“Serre” implies “Artin”).
Me? Well, I can do a little better, and prove many new cases of the strong Artin conjecture for totally odd representations

$$\rho : G_F \to GL_2(\mathbb{C})$$

of the absolute Galois group $G_F$ of a totally real field $F$.

Remark. It does not seem possible to generalise Khare-Wintenberger (while straightforward to check Khare’s “Serre” $\Rightarrow$ “Artin” in the Hilbert case); and “Serre” + (“Serre” $\Rightarrow$ “Artin”) to prove the strong Artin conjecture is probably not a good idea.
**Theorem 1** (S, 2010) Let $F$ be a totally real field. Assume that 5 splits completely in $F$. Let $\rho : G_F \to GL_2(\mathbb{C})$ be a continuous, totally odd, and icosahedral representation of $G_F = \text{Gal}(F^{\text{alg}}/F)$.

Suppose that, for every place $v | 5$, the projective image of the decomposition group $G_v$ has order 2 and the corresponding quadratic extension in $F_v^{\text{alg}}$ of $F_v$ is not $\mathbb{Q}_5(\sqrt{5})$.

Then the strong Artin conjecture for $\rho$ holds.

Remark. Instead of the conditions above, I can prove the strong Artin conjecture assuming 2 splits completely in $F$ (and slightly different condition at 2).

Remark. In fact, I can even do this for the totally ramified case...
Remark. And I have absolutely no idea how to remove the condition at 5.
**Theorem 2** *(S, 2010)* Let $F$ be a totally real field. Suppose that a prime $p$ is unramified in $F$. Let $L$ be a finite extension of $\mathbb{Q}_p$ with maximal ideal $m_L$. Let $\rho : G_F \to GL_2(O_L)$ be a representation of $G_F$ which

(1) ramifies at only finite many places of $F$;

(2) the restriction to $G_F(\zeta_p)$ of $\bar{\rho} = (\rho \mod m_L)$ is absolutely irreducible and $\bar{\rho}$ arises from a Hilbert modular form;

(3) for any $v | p$, $\rho$ is “nearly ordinary at $v$”, i.e., the restriction $\rho|G_v$ to the decomposition group $G_v$ is of the form

$$\rho|G_v \simeq \begin{pmatrix} \alpha_v & * \\ 0 & \beta_v \end{pmatrix}$$

such that

(3-1) $\alpha_v|I_v$ and $\beta_v|I_v$ are finite when restricted to the inertia group at $v$, 

and

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\((3-2) \ (\alpha_v \mod \mathfrak{m}_L) \neq (\beta_v \mod \mathfrak{m}_L)\).

Then there exists an overconvergent Hilbert modular form of weight 1 and the twist of its associated Galois representation is \(\rho\).

In particular, if one furthermore assumes

(1) \(p\) splits completely in \(F\)

and

(2) \(\rho\) is split at all \(v|p\), i.e., \(\rho|_{G_v}\) is diagonalisable,

then \(\rho\) arises from a Hilbert modular form \(f\) of weight 1, and there exists an embedding \(\mathbb{Q}(\{a(n,f)\}) \hookrightarrow L\) and, when followed by any embedding of \(L\) into \(\mathbb{C}\), the strong Artin conjecture holds.
So the part (2) is settled. How about (1)? Well, this is the part I’d like to talk to you about today.

**Theorem 3** *(S, 2011)* Let $F$ be a totally real field. Assume that $F$ is linearly disjoint from $\mathbb{Q}(\sqrt{5})$ (e.g. 5 is unramified in $F$). Let $\bar{\rho} : G_F \rightarrow GL_2(\overline{F}_5)$ be a continuous and totally odd representation of $G_F$. Suppose that

(1) $\bar{\rho}$ has projective image $A_5$;

(2) the projective image of the decomposition group $G_v$ for every $v|5$ has order 2, and the quadratic extension of $F_v$ corresponding to the projective image is not $F_v(\sqrt{5})$.

Then $\bar{\rho}$ is modular.
Proof.

“POTENTIALLY” LIFTING ICOSAHEDRAL REPRESENTATIONS

Find a totally real soluble extension $F_1$ of $F$ such that $\bar{\rho}_1 := \bar{\rho}|_{G_{F_1}} : G_{F_1} \rightarrow GL_2(\mathbb{F}_5)$ has determinant the cyclotomic character.

So $\bar{\rho}_1$ “looks like” it arises from an elliptic curve.

To do this, observe that the obstruction for lifting $\bar{\rho} : G_F \rightarrow A_5 \simeq PSL_2(\mathbb{F}_5)$ to a homomorphism $G_F \rightarrow SL_2(\mathbb{F}_5)$ lies in $H^2(G_F, \{\pm 1\})$.

Since

$$H^2(G_F, \{\pm 1\}) \xrightarrow{\text{res}} \bigoplus_v H^2(G_{F_v}, \{\pm 1\}),$$

choose (by CFT) a bi-quadratic totally real extension $F_1$ of $F$ in which the finite places $v$ in $F$ where the local obstructions are non-trivial, do not split completely.
At the infinite places, the local obstructions remain non-trivial.

On the other hand, the obstruction for lifting $G_F \to \{\pm 1\}$ to a character $G_F(\sqrt{5}) \to \mathbb{F}_5^\times$ with square mod 5 cyclotomic character lies in $H^2(G_F,\{\pm 1\})$, and non-trivial exactly at the infinite places.

The obstructions for the two lifting problems (which are exactly at the infinite places) cancel out each other!
A MODULI SPACE OF MOTIVES

Let $F_2$ be the Galois closure over $\mathbb{Q}$ of an extension of $F_1$ in which $\sqrt{5}$ splits completely.

Find an elliptic curve $E$ over a finite soluble extension $F_2$ of $F_1$ such that

(1) $\bar{\rho}_{E,3} : G_{F_2} \to GL(E[3]) \simeq GL_2(\mathbb{F}_3)$ is surjective;

(2) $\bar{\rho}_{E,3}|_{G_{F_2}(\sqrt{-3})}$ is absolutely irreducible

(3) $E$ has (potentially) good ordinary reduction at every $v|5$

(4) $\bar{\rho}_{E,5} : G_{F_2} \to GL(E[5]) \simeq GL_2(\mathbb{F}_5)$ is isomorphic to $\bar{\rho}_2 := \bar{\rho}_1|_{G_{F_2}}$ (up to twist by a character).

To do this, consider a moduli space $Y_{\bar{\rho}_2}$ of elliptic curves over $F_2$ whose 5-torsions are isomorphic to $\bar{\rho}_2$. There are infinitely many $F_2$-rational points. Find a $F_2$-point of $Y_{\bar{\rho}_2}$. 
corresponding to an elliptic curve over $F_2$ which has (potentially) good ordinary reduction at every $v|5$. Then there is a $F_2$-point, which is close (for the 5-adic topology) to the point and which is not in the image (finite many points) of the $F_2$-points of $Y_{\bar{\rho}_2,0}(3) = \{(E,C)\}/\simeq$ nor in the image (finitely many points) of $F_2$-points of $Y_{\bar{\rho}_2,\text{split}}(3) = \{(E,\{C,D\})\}/\simeq$. 
POTENTIAL AUTOMORPHY

By Langlands-Tunnell, $E[3]$ is modular by (1). It follows from Kisin’s MLT ($p = 3$), $\rho_{E,3}$ is modular (see (2)). By Falting’s isogeny theorem, $E$ is modular. In particular, $\overline{\rho}_{E,5}$ is modular and therefore, by (3), $\overline{\rho}_2$ is modular. By a generalisation of Taylor’s argument in “Artin II”, there is a lifting $\rho : G_F \to GL_2(\mathbb{Z}_5)$ of $\overline{\rho}$ such that $\rho|_{G_{F_2}}$ is a lifting of $\overline{\rho}_2$. By Skinner-Wiles ($p = 5$), $\rho|_{G_{F_2}}$ is modular. Since $F_2$ is a totally real soluble extension of $F$, by decent, $\rho$ is modular. In which case $\overline{\rho}$ is modular. \qed