# A conjecture of Fontaine-Mazur and weight one forms over totally real fields

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# 1 Introduction

Let $p$ be a rational prime. A conjecture of Fontaine-Mazur [37] asserts:

**Conjecture 1.** *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation of the absolute Galois group of a number field $F$. If $\rho$ is unramified at all but finitely many places of $F$ and is potentially semi-stable at every finite place of $F$ above $p$, then $\rho$ is 'geometric'.*

This specialises to the following conjecture of Fontaine-Mazur:

**Conjecture 2.** *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation of the absolute Galois group of a number field $F$. If $\rho$ is unramified at all but finitely many places of $F$ and is potentially unramified (i.e the image of the inertia subgroup is finite) at every finite place of $F$ above $p$, then $\rho$ has finite image.*

This paper proves many new cases of Conjecture 2 when $n = 2$, $F$ is a totally real number field, and $\rho$ is assumed to be totally odd (i.e. the image by $\rho$ of complex conjugation with respect to every embedding $F \to P$ has determinant $-1$) and the associated mod $p$ representation $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F}_p)$ is modular. More precisely, we prove the following: Let $L$ be a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, maximal ideal $\lambda$ and residue field $\mathbb{F} = \mathcal{O}/\lambda$.

**Theorem 3.** *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation of the absolute Galois group of a totally real field $F$ such that*

- *$\rho$ is totally odd,*
- *the image of the inertia subgroup at every finite place of $F$ above $p$ is finite.*
- *$\overline{\rho} = (\rho \bmod \lambda)$ is modular– there exists a cuspidal automorphic representation $\Pi$ of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ whose associated $p$-adic Galois representation is isomorphic to $\overline{\rho}$– when $\overline{\rho}$ is absolutely irreducible with insoluble image; and suppose furthermore than $\Pi$ is ordinary at every place of $F$ above $p$ when $p = 2$ and $\overline{\rho}$ is unramified (i.e. trivial) at every infinite place of $F$.*
- *The semi-simplification of $\overline{\rho}$ is not scalar, i.e. not twist-equivalent to the trivial representation.*

*Then there exists a holomorphic modular eigenform of parallel weight 1 on $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ whose associated $p$-adic representation of $\mathrm{Gal}(\overline{F}/F)$ is isomorphic to $\rho$. In particular, $\rho$ has finite image.*

The finiteness of Galois representations associated to Hilbert modular eigenforms of parallel weight 1 is a well-known result of Deligne-Serre [32] ($F = \mathbb{Q}$) and Rogawski-Tunnell [73] (general $F$).

We hasten to remark that the last/fourth condition follows from the third assumption (the oddness of $\overline{\rho}$) when $p > 2$.

A case of the main theorem is established in [75], when $p \geq 5$ and $\overline{\rho}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$ (and if $p = 5$ and the projective image of $\overline{\rho}$ is $\mathrm{PGL}_2(\mathbb{F}_5)$, it is furthermore assumed that the kernel of the projective representation of $\overline{\rho}$ does not fix $F(\zeta_5)$); and, as a corollary, Artin's conjecture for totally odd continuous representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{C})$ is proved completely.

By assumption, after replacing $F$ by its finite totally real soluble extension if necessary, it is possible to assume $\rho$ is $p$-ordinary (i.e. reducible at every place of $F$ above $p$), and we prove that $\rho$ is pro-modular, i.e., arises from a $p$-ordinary $p$-adic modular eigenform. To this end, we directly compare $p$-adic families, '$R$' of $p$-ordinary Galois representations and '$T$' of $p$-ordinary modular forms (whether $p$ is odd or not), without recourse to unitary groups over CM fields (hence our pro-modularity results do not follow from Thorne's [89] 'by functoriality'). In fact, our overall approach to construct weight one forms is crucially dependent on what we know about *all* Hecke eigenvalues (hence $q$-expansion coefficients by 'multiplicity one') of $p$-adic eigenforms (especially those generalised eigenvalues at places above $p$), and we know of no other route than to establish an '$R = T$' theorem. The Calegari-Geraghty variant [18] of the Taylor-Wiles argument may well allow us to circumvent some of the issues arising from the 'multiplicity one problem' (by directly comparing deformations of Galois representations that are potentially unramfied at $p$ and the coherent cohomology complex of parallel weight one Hilbert modular forms) but this alternative approach is still contingent upon several outstanding conjectures about the local-global compatibility of automorphic Galois representations.

We establish the aforementioned pro-modularity via finding a co-dimension 1 prime $\Gamma$ of $R$ containing $p$, such that the specialisation $\rho_\Gamma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F}[[\pi]])$ of the universal $p$-ordinary deformation over $R$ of $\overline{\rho}$ at $\Gamma$ is irreducible *and* is not dihedral (i.e. not induced from a quadratic extension of $F$). The irreducible $\rho_\Gamma$, with 'large image', over the discrete valuation ring $\mathbb{F}[[\pi]]$, instead of $\overline{\rho}$, is then used to pull off an analogue of the Taylor-Wiles argument. In finding a such prime $\Gamma$, while we allow ourselves to trade $F$ for its finite totally real soluble extensions where convenient, ascertaining $\rho_\Gamma$ is irreducible and non-dihedral requires us to tread carefully.

Suppose that $\overline{\rho}$ is irreducible with soluble image and is induced from a quadratic extension $E$ of $F$. On one hand, assuming that $E$ is *not* an imaginary quadratic extension of $F$ in which every place of $F$ above $p$ splits completely, it follows without further expenditure of effort that $\rho_\Gamma$ is irreducible and non-dihedral. This is the approach taken by Skinner [79] ($p > 2$) and Allen [1] ($p = 2$), though the latter works under the further assumption in which determinants are fixed as in Khare-Wintenberger [56]. On the other hand, when $\overline{\rho}$ is induced from an imaginary quadratic extension $E$ of $F$ in which every place of $F$ above $p$ splits completely, it is a priori possible for $\rho_\Gamma$ to be induced from the CM extension. To overcome this problem, we make appeal to the pseudo-representation theory, due mostly to Bellaïche and Chenevier [9], to observe that if the universal deformation over $R$ is not dihedral, i.e. not induced from a character of $\mathrm{Gal}(\overline{F}/F^+)$ for any quadratic extension $F^+$ of $F$ (resp. is dihedral and induced from a character of $\mathrm{Gal}(\overline{F}/E)$), then one can find a non-dihedral prime over an infinitesimal deformation of $\overline{\rho}$, as a precursor to conjuring up $\Gamma$ as above (resp. one can instead identify $R$ directly with the quotient of $T$ arising from $E$-CM forms). This is a variation of the well-known observation (see work of Ghate, Dimitrov, Wiese and others) that non-CM $p$-ordinary families of (Hilbert) modular forms almost never intersect with CM $p$-ordinary families (even at weight one).

The question about whether it is possible at all to find $\Gamma$ with irreducible $\rho_\Gamma$ is prevalent when $\overline{\rho}$

is reducible. In this case, it is not unreasonable to expect that the 'reducible quotient' $R_\Delta$ of the $p$-ordinary deformation ring $R$, where corresponding representations are (globally) reducible, define irreducible components of $R$. In fact, we do not know a priori that $R$ is equi-dimensional! These make our search for an 'irreducible' prime $\Gamma$ (which should inevitably lie in $R - R_\Delta$) more difficult. For example, when $F = \mathbb{Q}$, the reducible ('Eisenstein') locus does define irreducible components of the dimension equal to that of the irreducible ('cuspidal') components.

Skinner & Wiles were the first to tackle this issue in [80], and their work has been vastly generalised by Thorne [89] and his collaborators [2] in arbitrary dimension. To elaborate, let $1 + \gamma_F$ be the $\mathbb{Z}_p$-rank of the $p$-adic closure in $(\mathscr{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ of the group $\mathscr{O}_{F,+}^\times$ of totally positive units in the ring $\mathscr{O}_F$ of integers of $F$. The Leopoldt conjecture asserts that $\gamma_F$ should be 0. Assuming that $\overline{\rho}$ is reducible with its semi-simplification of the form $1 \oplus \overline{\chi}$ say, we know, on one one hand, $R_\Delta$ has dimension bounded above by $r_F = 1 + 2(1 + \gamma_F) + \dim_{\mathbb{F}} \mathrm{Ext}^1(1, \overline{\chi})$, while, on the other hand, the dimension of $R$ is bounded below by $s_F = 1 + [F : \mathbb{Q}] + \gamma_F$ in terms dimensions of local versal deformation rings at ramified places. It is reasonable to expect $r_F \leq s_F$, but there is nothing conclusive to suggest that the inequality would always have to be strict! This is mostly to do with the fact that it is hard to get one's hands on $\dim_{\mathbb{F}} \mathrm{Ext}^1(1, \overline{\chi})$. To circumvent the issue, Skinner-Wiles [80] instead makes appeal to Washington's result [94] (resp. Waldschmidt's [92]) about $\ell$-adic Iwasawa invariants (for $\ell$ not equal to $p$) to bound $\dim_{\mathbb{F}} \mathrm{Ext}^1(1, \overline{\chi})$ (resp. $1 + \gamma_F$), and manages to find a finite soluble totally real extension $F'$ of $F$ of a sufficiently large $\ell$-power degree for which the strict inequality $r_{F'} < s_{F'}$ holds[1]. This 'relative smallness' of $R'_\Delta$ with respect to $R'$ consequently allows [80] to find a prime $\Gamma$ in $R'$ not lying in $R'_\Delta$, i.e. an irreducible $\Gamma$. The drawback of this argument, however, is that Washington's result requires $F'$ to be abelian over $\mathbb{Q}$ and this significantly qualifies the pro-modularity theorem of [80]. L. Pan [67] follows the strategy of Skinner-Wiles, and has similar, but more general, results about modularity of $p$-adic representations $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which are potentially semi-stable at $p$, but for an abelian extension $F$ over $\mathbb{Q}$ in which $p$ splits completely.

In this paper, we remove this 'abelian condition' entirely. The gist of our argument is that, instead of trying to achieve $r_F < s_F$, we prove pro-modularity separately over Spec $R_\Delta$ and each affine open covering of the complement Spec $R -$ Spec $R_\Delta$; for the former, a theory of Eisenstein series achieves this, while for the latter, the irreducibility of $\Gamma$ comes for free (!) even though $\overline{\rho}$ is reducible, and an argument similar to the one in the generic case attains pro-modularity. To make this part of the argument work, we make essential use of the pseudo-deformation theory [91] developed by Chenevier [21], Bellaiche-Chenevier [9] and Wake–Wang-Erickson [91]. To establish that the pull-back of the universal Galois representation over Spec $R$ to any covering of Spec $R -$ Spec $R_\Delta$ is indeed irreducible, it is necessary for us to know that it is of 'Generalised Matrix Algebra' type and, for this reason, $\overline{\rho}$ (and its restriction to a finite soluble totally real field extension of $F$) is assumed to be non-trivial. This can be attained by maintaining $\overline{\rho}$ to be totally odd if $p > 2$ but not necessarily so when $p = 2$.

The additional assumption in the third in the case of $p = 2$ can be removed if one can prove a result of the form '$\overline{\rho}$ is modular $\Rightarrow \overline{\rho}$ is $p$-ordinary modular (up to a finite totally real soluble base change)' as in [87]. This is a problem of different nature and we hope to come back to it separately in the future.

[1]Successful distillation of the argument in [80] initiated this paper.

In a forthcoming work, we hope to address the case when $\overline{\rho}$ is twist-equivalent to the trivial representation.

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# 2 Deformations of Galois representations

Fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ once for all. Choose an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ once for all.

Let $F$ be a totally real field, $\mathbb{F}$ be a field of characteristic $p > 0$, and $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ be a totally odd (i.e. the image of complex conjugation with respect to every embedding of $F$ into $P$ is non-trivial), continuous representation of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of $F$.

Let $\chi_{\mathrm{cyc}}$ denote the $p$-adic cyclotomic character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_p^\times$ and $\overline{\chi}_{\mathrm{cyc}}$ denote its mod $p$ variant.

Let $L$ denote a finite extension of $\mathbb{Q}_p$ containing the image of every embedding $F \to \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and let $\mathscr{O}$ denote its ring of integers with residue field $\mathbb{F}$. Let $\lambda$ denote a uniformiser we fix throughout the paper.

For every place v of $F$, let $F_{\mathrm{v}}$ denote the completion of $F$ at v, $\mathscr{O}_{F_{\mathrm{v}}}$ denote the ring of integers and $\mathbb{F}_{\mathrm{v}}$ the residue field at v. Let $\pi_{\mathrm{v}}$ denote a uniformiser of $\mathscr{O}_{F_{\mathrm{v}}}$. Let $D_{\mathrm{v}} \simeq \mathrm{Gal}(\overline{F}_{\mathrm{v}}/F_{\mathrm{v}})$ denote the decomposition subgroup at v and $I_{\mathrm{v}}$ the inertia subgroup at v.

As in [42] (and [22]), we 'normalise' the local Langlands correspondence in such a way that if $\Pi$ is an irreducible admissible representation of $\mathrm{GL}_2(K)$ for a finite extension $K$ of $\mathbb{Q}_p$, then the corresponding Weil-Deligne representation is the one associated to $\Pi^\vee \otimes |\ |^{-1/2}$ by Harris-Taylor's local Langlands correspondence.

Let $\mathscr{C}$ denote the category whose objects are artinian local $\mathscr{O}$-algebras $R$ for which the structure map $\mathscr{O} \to R$ induces an isomorphism on residue fields; and let $\hat{\mathscr{C}}$ denote the full subcategory of the category whose objects are topological $\mathscr{O}$-algebras which are limits of objects in $\mathscr{C}$. The morphisms of $\mathscr{C}$ and $\hat{\mathscr{C}}$ are continuous homomorphisms of $\mathscr{O}$-algebras which induce isomorphisms on the residue fields.

For every place v above $p$, we identify, via the local Artin map $\mathrm{Art}_{\mathrm{v}}$, once for all:

- the pro-$p$-part $1 + \pi_{\mathrm{v}}\mathscr{O}_{F_{\mathrm{v}}}$ of $\mathscr{O}_{F_{\mathrm{v}}}^\times$
- the image $\Delta_{\mathrm{v}}$ of the inertia subgroup at v in the pro-$p$ completion of the maximal abelian quotient of the decomposition subgroup $D_{\mathrm{v}}$ at v.

Let $\Delta_p = \prod_{\mathrm{v}} \Delta_{\mathrm{v}}$ and let $\Lambda_p$ denote Iwasawa (group) algebra $\mathscr{O}[[\Delta_p]]$ (of relative dimension $\sum_{\mathrm{v}}[F_{\mathrm{v}} : \mathbb{Q}_p] = [F : \mathbb{Q}]$ over $\mathscr{O}$). On the other hand, let $\Delta^p$ denote the pro-$p$ completion of the Galois group of the maximal extension of $F$ unramified outside a finite set $S$ of places in $F$ containing the set $S_p$ of places of $F$ above $p$ and the set $S_\infty$ of infinite places of $F$, and let $\Lambda^p$ denote the Iwasawa group algebra $\mathscr{O}[[\Delta^p]]$ (of relative dimension $1+\gamma_F = \mathrm{rk}_{\mathbb{Z}_p}(\mathscr{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\overline{\mathscr{O}_F^\times}$, where $\gamma_F$ denotes the Leopoldt defect for $(F,p)$ which is conjectured to be 0). Let

$$\Lambda = \Lambda_p \hat{\otimes} \Lambda^p;$$

it is of dimension $1+[F:\mathbb{Q}]+\gamma_F$ over $\mathscr{O}$.

There is a universal one-dimensional deformation

$$\phi = \prod_{\mathrm{v}} \phi_{\mathrm{v}} : \Delta_p = \prod_{\mathrm{v}} \Delta_{\mathrm{v}} \to \Lambda_p$$

of the trivial representation $\Delta_p \to \mathbb{F}^\times$ and a universal one-dimensional deformation

$$\phi' : \Delta^p \to \Lambda^p$$

of the trivial representation $\Delta^p \to \mathbb{F}^\times$. We often see via

$$(\phi, \phi') \mapsto (\chi, \chi') = (\phi, \prod_{\mathrm{v}} \phi_{\mathrm{v}}^{-1} \phi'|_{\Delta_{\mathrm{v}}})$$

that $\Lambda = \Lambda_p \hat{\otimes} \Lambda^p$ also parameterises the pairs $(\chi, \chi')$ of one-dimensional deformations of $\Delta_p \to \mathbb{F}^\times$ over local noetherian $\mathscr{O}$-algebras with residue field $\mathbb{F}$ such that their product factors through the $p$-adic completion $\overline{\mathscr{O}_F^\times} \cap \Delta_p$.

A character of $I_{\mathrm{v}}$ (or $\Delta_{\mathrm{v}}$) is said to be algebraic of weight $r_{\mathrm{v}} = (r_\tau) \in \mathbb{Z}^{\mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathrm{v}}, L)}$ if it is given by $\prod_\tau (\tau \circ \mathrm{Art}_{\mathrm{v}}^{-1})^{-r_\tau}$. A pair $(\chi, \chi')$ of one-dimensional deformations as above is said to be algebraic if there exists a pair $(k=(k_{\mathrm{v}}), \ell=(\ell_{\mathrm{v}}))$ of $[F:\mathbb{Q}]$-tuple of integers with $k_{\mathrm{v}} = \sum_\tau k_\tau \tau$ and $\ell_{\mathrm{v}} = \sum_\tau \ell_\tau \tau$ and $k_\tau \geq 1$ and $\ell_\tau \geq 0$ for every $\tau$ in $\mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathrm{v}}, L)$, such that $\chi_{\mathrm{v}}$ (resp. $\chi'_{\mathrm{v}}$) is algebraic of weight $\ell_{\mathrm{v}}$ (resp. $\ell_{\mathrm{v}} + k_{\mathrm{v}} - 1$) for every v in $S_p$.

Give a finite set of places of $F$ as above, Let $F_S$ denote the maximal extension of $F$ unramified outside $S$. Let $\Sigma = (S, T, \{L_{\mathrm{v}}\})$ denote a deformation data consisting of $S$, a subset $T$ of 'framed places' and $L_{\mathrm{v}} \subset H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})$ for every v in $S$ which defines a local deformation problem at v with corresponding ideal $I_{\mathrm{v}}^\square$ of $R_{\mathrm{v}}^\square$, with $N_{\mathrm{v}} = H^0(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})$ for every finite place v in $S-T$.

$$\begin{array}{lllll}
0 \to & H^0_\Sigma(F, \mathrm{ad}\,\overline{\rho}) & \to H^0(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}) & \to & \bigoplus_T H^0(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \oplus \bigoplus_{\mathrm{v}\in S-T} H^0(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})/N_{\mathrm{v}} \\
\to & H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho}) & \to H^1(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}) & \to & \bigoplus_T H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \oplus \bigoplus_{\mathrm{v}\in S-T} H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})/L_{\mathrm{v}} \\
\to & H^2_\Sigma(F, \mathrm{ad}\,\overline{\rho}) & \to H^2(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}) & \to & \bigoplus_T H^2(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \oplus \bigoplus_{\mathrm{v}\in S-T} H^2(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \\
\to & H^2_\Sigma(F, \mathrm{ad}\,\overline{\rho}) & \to H^3(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}) & \to & \cdots
\end{array}$$

Given a deformation data $\Sigma$ as above, we let $R_\Sigma^\square$ the universal ring for $T$-framed deformations of type $\Sigma$ in the sense of Definition 2.2.7 in [22]. Let $H^r_\Sigma(F, \mathrm{ad}\,\overline{\rho})$ denote the $r$-th cohomology of the complex defined by $\Sigma$ as in [22] (See Chapter 2 in [75]).

As $\Sigma$ reads ramification of classes in $H^1(\mathrm{Gal}(\overline{F}/F), \mathrm{ad}\,\overline{\rho})$, by $H^r_\Sigma(F, \mathrm{ad}\,\overline{\rho})$, we really mean $H^r_\Sigma(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho})$. We often write $H^1(F_S, \mathrm{ad}\,\overline{\rho})$ for $H^1(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho})$ and do it similarly with $\mathrm{ad}\,\overline{\rho}(1)$ in place of $\mathrm{ad}\,\overline{\rho}$.

We let $L_v^\perp$ denote the annihilator of $L_v$ in $H^1(D_v, \mathrm{ad}\,\overline{\rho}(1))$ induced by the pairing $\mathrm{ad}\,\overline{\rho} \times \mathrm{ad}\,\overline{\rho}(1) \to \mathbb{F}(1)$ and we let

$$H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}) = \ker \left( H^1(F_S, \mathrm{ad}\,\overline{\rho}(1)) \to \bigoplus_{S-T} H^1(D_v, \mathrm{ad}\,\overline{\rho}(1))/L_v^\perp \right)$$

Let $A_\Sigma^\square$ denote the completed tensor product over $\mathcal{O}$

$$\hat{\bigotimes}_{v\in T} R_v^\square / I_v^\square$$

of the quotient $R_v^\square/I_v^\square$ of $R_v^\square$ by the ideal $I_v^\square$ defined by $L_v$, as v ranges over $T$. Let $R^\square$ denote the formal power series ring in $4|T| - \dim_{\mathbb{F}} H^0(F_S, \mathrm{ad}\,\overline{\rho})$ variables with coefficients in $\mathcal{O}$, normalised such that

$$R_\Sigma^\square \simeq R_\Sigma \otimes R^\square.$$

## 2.1 Universal rings for global liftings

Suppose $n = 2$. Then

**Proposition 4.** $\dim_{\mathbb{F}} H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho})$ *equals*

$$\begin{aligned} & \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) + \dim_{\mathbb{F}} H^0_\Sigma(F, \mathrm{ad}\,\overline{\rho}) - \dim_{\mathbb{F}} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - \chi(F_S, \mathrm{ad}\,\overline{\rho}) \\ + & \textstyle\sum_{v\in S} \chi(D_v, \mathrm{ad}\,\overline{\rho}) + \sum_{v\in(S-T)} \dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} N_v \end{aligned}$$

We say $\overline{\rho}$ is *exceptional* if

- $p = 3$,
- $\overline{\rho}$ is induced from $\mathrm{Gal}(\overline{F}(\zeta_3)/F)$.

**Proposition 5.** *Suppose that*

- *For a place* v *in* $(S - T) \cap S_p$,

$$\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} N_v = \dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = [F_v : \mathbb{Q}_p]$$

- *For a place* v *in* $(S - T) \cap S_\infty$,

$$\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} N_v = -1.$$

*Then*

$$\dim_{\mathbb{F}} H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho}) = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 1 + \sum_{(S-T)-S_\infty} \left(\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho})\right) - [F : \mathbb{Q}]$$

*unless* $\overline{\rho}$ *is exceptional, in which case*

$$\dim_{\mathbb{F}} H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho}) = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 2 + \sum_{(S-T)-S_\infty} \left(\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho})\right) - [F : \mathbb{Q}].$$

*Proof*. This can be proved as in Proposition 5 in Section 2 of [75]. If $\overline{\rho}$ is not exception (resp. is exceptional), then $\dim H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) = \dim H^0(F_S, \mathrm{ad}^0\,\overline{\rho}(1)) + \dim H^0(F_S, \mathbb{F}(1)) = 0 + 1$ (resp. $\dim H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) = 2$).

Also $H^0_\Sigma(F, \mathrm{ad}\,\overline{\rho})$ is a subspace of the one-dimensional $\mathbb{F}$-vector space $H^0(F, \mathrm{ad}\,\overline{\rho})$, but the $N_v$ forces the dimension of the former to be strictly smaller than that of the latter, i.e., $H^0_\Sigma(F, \mathrm{ad}\,\overline{\rho}) = 0$ (whether $T = \varnothing$ or not!).

Furthermore, it follows from the global (resp. local) Euler characteristic formula (resp. formulae) that $\chi(F_S, \mathrm{ad}\,\overline{\rho}) = -2[F:\mathbb{Q}]$ (resp. $\sum_{v \in S} \chi(D_v, \mathrm{ad}\,\overline{\rho}) = \sum_{v \in S_p} -4[F_v : \mathbb{Q}_p] + \sum_{v \in S_\infty} 4 = 0$). □

**Remark**. $\dim_{\mathbb{F}} H^0(F, \mathrm{ad}\,\overline{\rho}) = 1$ whether $p$ is odd or not.

We now apply the formula above to

$$\Sigma_{Q,N} = (S \cup S_{Q,N}, T, \{L_v\}_{v \in S \cup S_{Q,N}})$$

to compute $\dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\overline{\rho})$, where $T \subset S - S_\infty$ and, for every v in $S_{Q,N} = ((S \cup S_{Q,N}) - T) - S_\infty$, the local deformations at v are defined such that

$$\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} N_v = \dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = 1$$

if $p > 2$ and such that $L_v = H^1(D_v, \mathrm{ad}\,\overline{\rho})$ and

$$\dim_{\mathbb{F}} L_v - \dim_{\mathbb{F}} N_v = \dim_{\mathbb{F}} H^1(D_v, \mathrm{ad}\,\overline{\rho}) - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = 2$$

if $p = 2$.

## 2.2 $S_\infty$

Following [10], for any infinite place v and a non-negative integer $\bullet$, we let $H^\bullet(D_v, \mathrm{ad}\,\overline{\rho})^*$ denote the image of $H^\bullet(D_v, \mathrm{ad}^0\overline{\rho})$ in $H^\bullet(D_v, \mathrm{ad}\,\overline{\rho})$. The versal odd deformation ring $R_v^{-1}$ (resp. the universal ring $R_v^{\square,-1}$ for odd liftings) gives rise to

- $N_v = \mathbf{N}_{\overline{F}_v/F_v}(\mathrm{ad}\,\overline{\rho}) \subset (\mathrm{ad}\,\overline{\rho})^{D_v}$ (so that $H^1(D_v, \mathrm{ad}\,\overline{\rho})/N_v$ is the zero-th Tate cohomology group,

- $L_v = H^1(D_v, \mathrm{ad}\,\overline{\rho})^*$.

We leave it as an exercise to check:

$$\dim_{\mathbb{F}} N_v = \begin{cases} 1 & \text{if } p > 2, \\ 1 & \text{if } p = 2 \text{ and } \overline{\rho}_v \text{ is non-trivial}, \\ 4 & \text{if } p = 2 \text{ and } \overline{\rho}_v \text{ is trivial}, \end{cases}$$

and

$$\dim_{\mathbb{F}} L_{\mathrm{v}} = \begin{cases} 0 & \text{if } p > 2, \\ 0 & \text{if } p = 2 \text{ and } \overline{\rho}_{\mathrm{v}} \text{ is non-trivial}, \\ 3 & \text{if } p = 2 \text{ and } \overline{\rho}_{\mathrm{v}} \text{ is trivial}, \end{cases}$$

for $\dim_{\mathbb{F}} H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})^* = \dim_{\mathbb{F}} H^1(D_{\mathrm{v}}, \mathrm{ad}^0\,\overline{\rho}) - \dim_{\mathbb{F}} \mathrm{Coker}(H^0(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \to H^0(D_{\mathrm{v}}, \mathbb{F}))$, where $\dim_{\mathbb{F}} H^1(D_{\mathrm{v}}, \mathrm{ad}^0\,\overline{\rho})$ can be computed by the archimedean Euler-Poincare characteristic and the local Tate duality

$$\dim_{\mathbb{F}} \mathrm{ad}^0\,\overline{\rho} - (\dim H^0(D_{\mathrm{v}}, \mathrm{ad}^0\,\overline{\rho}) + H^0(D_{\mathrm{v}}, \mathrm{Hom}_{\mathbb{F}}(\mathrm{ad}^0\,\overline{\rho}, \mathbb{F})(1)))$$

(when $p = 2$, $\mathrm{Hom}_{\mathbb{F}}(\mathrm{ad}^0\,\overline{\rho}, \mathbb{F})(1) \simeq \mathrm{ad}\,\overline{\rho}/\mathbb{F}$), and where

$$\dim_{\mathbb{F}} \mathrm{Coker}(H^0(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}) \to H^0(D_{\mathrm{v}}, \mathbb{F})) = \begin{cases} 0 & \text{if } p > 2, \\ 1 & \text{if } p = 2 \text{ and } \overline{\rho}_{\mathrm{v}} \text{ is non-trivial}, \\ 0 & \text{if } p = 2 \text{ and } \overline{\rho}_{\mathrm{v}} \text{ is trivial}. \end{cases}$$

## 2.3 $S_p$

Following [75], let $R_{\mathrm{v}}^{\square,\Delta}$, for every place v above $p$, denote the quotient of $R_{\mathrm{v}}^{\square} \hat{\otimes} \mathscr{O}[[\Delta_{\mathrm{v}} \times \Delta_{\mathrm{v}}]]$ by an ideal $I_{\mathrm{v}}^{\square,\Delta}$ parameterising $(\rho, \alpha(\phi), (\chi_1, \chi_2))$ where $\rho$ is a lifting of $\overline{\rho}_{\mathrm{v}}$, $\alpha(\phi)$ is a root of the polynomial $X^2 - \mathrm{tr}\,\rho(\phi) + \det \rho(\phi) = 0$ and $(\chi_1, \chi_2)$ is a pair of characters parameterised by $\mathscr{O}[[\Delta_{\mathrm{v}} \times \Delta_{\mathrm{v}}]]$ satisfying the conditions

(I) $\mathrm{tr}\,\rho(\sigma) = \chi_1(\sigma) + \chi_2(\sigma)$ for $\sigma$ in $I_{\mathrm{v}}$,

(II) $\mathrm{tr}\,\rho(\phi) = \alpha(\phi) + \beta(\phi)$ where $\beta(\phi)$ denotes $\det \rho(\phi)/\alpha(\phi)$,

(III) $\det(\rho(\phi) - \beta(\phi)) = 0$,

(IV) $1 + \det(\chi_2(\sigma)^{-1}\rho(\sigma)) = \mathrm{tr}\,(\chi_2(\sigma)^{-1}\rho(\sigma))$ for $\sigma$ in $I_{\mathrm{v}}$,

(V) $(\rho(\sigma) - \chi_2(\sigma))(\rho(\tau) - \chi_2(\tau)) = (\chi_1(\sigma) - \chi_2(\sigma))(\rho(\tau) - \chi_2(\tau))$ for $\sigma, \tau$ in $I_{\mathrm{v}}$,

(VI) $(\rho(\phi) - \alpha(\phi))(\rho(\sigma) - \chi_2(\sigma)) = (\beta(\phi) - \alpha(\phi))(\rho(\sigma) - \chi_2(\sigma))$ for $\sigma$ in $I_{\mathrm{v}}$.

We firstly establish that $R_{\mathrm{v}}^{\square,\Delta}/\lambda$ is Cohen-Macaulay and reduced. To see this, we follow the proof of Proposition 5 in [75] and let $S_{\mathrm{v}}^{\square,\Delta}/\lambda$ (resp. $S_{\mathrm{v}}^{\square,\Delta}[1/\lambda]$) denote the quotient of the ring of polynomials over $\mathbb{F}$ (resp. $L$) in $5[F_{\mathrm{v}} : \mathbb{Q}_p] + 5$ variables, by the ideal generated by the 2-by-2 minors of the $2 \times (2[F_{\mathrm{v}} : \mathbb{Q}_p] + 2)$ matrix. It follows from Theorem 2.7 (resp. Theorem 2.11) in [14] that $S_{\mathrm{v}}^{\square,\Delta}/\lambda$ and $S_{\mathrm{v}}^{\square,\Delta}[1/\lambda]$ are Cohen-Macaulay (resp. reduced). In fact, these rings are often known as determinantal rings and are known to be normal domains.

Any lifting of $\overline{\rho}_{\mathrm{v}}$ parametrised by $R_{\mathrm{v}}^{\square,\Delta}$ factors through the Galois group $\mathbb{G}_{\mathrm{v}}$ of the maximal pro-$p$ extension of $F_{\mathrm{v}}$ whose inertia subgroup $\mathbb{I}_{\mathrm{v}}$ is abelian of exponent $p$. The structure (i.e. generators and relations) of $\mathbb{G}_{\mathrm{v}}$ is given, for example, in Chapter VII, Section 5, of [65] or Section 5 of [60], while the quotient $\mathbb{G}_{\mathrm{v}}/\mathbb{I}_{\mathrm{v}}$ is topologically generated by a Frobenius lift. Since $\mathbb{I}_{\mathrm{v}}$ is abelian, the 'relations' boil down to one single equation in the case when $F_{\mathrm{v}}$ contains a $p$-th root of unity, leaving $\mathbb{I}_{\mathrm{v}}$ freely generated by $[F_{\mathrm{v}} : \mathbb{Q}_p]$ elements (whether $p$ is odd or not). It therefore follows that the map $R_v^{\square,\Delta}/\lambda \to S_v^{\square,\Delta}/\lambda$, defined explicitly in the proof of Proposition 5 in [75], is an isomorphism.

It follows from Proposition 2.2.1 in [82] that $R_v^{\square,\Delta}$ is flat over $\mathcal{O}$. Proposition 2.3.1 in [82] (resp. Theorem 2.1.3 in [13]) then proves $R_v^{\square,\Delta}$ is reduced (resp. Cohen-Macaulay).

Since $R_v^{\square,\Delta}[1/\pi]$ is isomorphic to a completion of $S_v^{\square,\Delta}[1/\pi]$ and the latter is a normal domain, it follows from the Zariski Main Theorem that $R_v^{\square,\Delta}[1/\pi]$ (hence $R_v^{\square,\Delta}$) is a normal domain. As a result, Spec $R_v^{\square,\Delta}/\Gamma$ is geometrically irreducible for a minimal ideal $\Gamma$ of $\mathcal{O}[[\Delta_v \times \Delta_v]]$.

**Lemma 6.** *Let $(\rho : D_v \to \mathrm{GL}_2(R), \alpha(\phi), (\chi_1, \chi_2))$ be a point of* Spec $R_v^{\square,\Delta}$ *defined over an artinian local $\mathcal{O}$-algebra $R$ (with residue field $\mathbb{F}$). Suppose that $\dfrac{\chi_1}{\chi_2}$ is neither trivial nor the cyclotomic character. Then the localisation of $R_v^{\square,\Delta}$ at the prime ideal defined by $\rho$ is regular.*

*Proof.* Since the completion of $S_v^{\square,\Delta}$ is $R_v^{\square,\Delta}$, it suffices to establish that the localisation of $S_v^{\square,\Delta}$ at $\rho$ is regular. To this end, we apply Theorem 2.6 in [14] to $S_v^{\square,\Delta}$. It remains to show that the prime ideal corresponding to $\rho$ does not contain the ideal generated by the $1 \times 1$ minors, i.e. the ideal generated over $\mathcal{O}$ by the $5[F_v : \mathbb{Q}_p] + 5$ variables. However, if it did contain the ideal, it follows that $\chi_1$ and $\chi_2$ would have to be equal (see the proof of Proposition 5 in [75]). □

## 2.4 $S_{Q,N}$

Let $N$ be an integer, assumed merely to be $> 1$ if $p > 2$ and assumed to be sufficiently large if $p = 2$. For v in $S_{Q,N}$ we consider the 'Taylor-Wiles' primes. Suppose that v satisfies $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v} \equiv 1 \bmod p^N$. Suppose that $\overline{\rho}_v$ is unramified, and is the direct sum of (unramified) characters $\overline{\chi}_{v,1}, \overline{\chi}_{v,2} : D_v \to \mathbb{F}^\times$ such that $\overline{\chi}_{v,1}(\mathrm{Frob}_v)$ and $\overline{\chi}_{v,2}(\mathrm{Frob}_v)$ are distinct. Then it follows from Hensel's lemma (see Lemma 2.44 in [25]) that every lifting $\rho$ of $\overline{\rho}_v$ is of the form $\rho = \chi_{v,1} \oplus \chi_{v,2}$ of $\overline{\rho}_v$ such that $\chi_{v,1}$ (resp. $\chi_{v,2}$) lifts $\overline{\chi}_{v,1}$ (resp. $\overline{\chi}_{v,2}$) and $\chi_{v,2}$ is unramified. For a such v, we define the subspace $L_v \subset H^1(D_v, \mathrm{ad}\,\overline{\rho}) = H^1(D_v, \mathrm{ad}\,\overline{\chi}_{v,1}) \oplus H^1(D_v, \mathrm{ad}\,\overline{\chi}_{v,2})$ to be

$$L_v = H^1(D_v, \mathrm{ad}\,\overline{\chi}_{v,1}) \oplus \ker\left(H^1(D_v, \mathrm{ad}\,\overline{\chi}_{v,2}) \to H^1(I_v, \mathrm{ad}\,\overline{\chi}_{v,2})\right).$$

Existence of a set $S_{Q,N}$ of such 'Taylor-Wiles primes' will be proved case-by-case in the following.

## 2.5 $S_R$ and $S_L$

Let v be a finite place of $F$ not dividing $p$ such that $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v} \equiv 1 \bmod p$. Suppose that

$$\overline{\rho}_v : D_v \to \mathrm{GL}_2(\mathbb{F})$$

is trivial. Let $\zeta = (\zeta_1, \zeta_2)$ be a pair of characters $D_v \to \mathcal{O}^\times$ such that the reduction $\overline{\zeta}_1 : D_v \to \mathcal{O}^\times \to \mathbb{F}^\times$ (resp. $\overline{\zeta}_1 : D_v \to \mathcal{O}^\times \to \mathbb{F}^\times$) of $\zeta_1$ (resp. $\zeta_2$) is trivial.

We may and will define the quotient $R_v^\square/I_v^{\square,\zeta}$ to be the maximally reduced and $\mathcal{O}$-flat quotient of $R_v^\square$ such that, for any finite extension $K$ of $L$, $\mathrm{Hom}_L(K, R_v/I_v^{\square,\zeta})$ is in bijection with liftings $D_v \to \mathrm{GL}_2(R_v^\square) \to \mathrm{GL}_2(K)$ of the trivial representation of $\overline{\rho}_v$ such that the semi-simplified restriction to the inertia subgroup at v is given by $\begin{pmatrix}\zeta_1 & 0\\ 0 & \zeta_2\end{pmatrix}$.

When $\zeta$ is trivial, i.e., both $\zeta_1$ and $\zeta_2$ are trivial, let $I_v^{\square,\mathrm{St}}$ denote the ideal of $R_v^\square$ containing $I_v^{\square,\zeta}$ such that $R_v^\square/I^{\square,\mathrm{St}}$ is reduced and $\mathcal{O}$-flat and $\mathrm{Hom}_L(K, R_v^\square/I_v^{\square,\mathrm{St}})$ parameterises the liftings

$\rho : D_{\mathrm{v}} \to \mathrm{GL}_2(K)$ of trivial representation $D_{\mathrm{v}} \to \mathrm{GL}_2(\mathbb{F})$ which has inertial type $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, N)$ for a non-trivial 2-by2 matrix $N$ of $\mathrm{GL}_2(L)$ in the sense of [78]. Note that the non-triviality of $N$ forces the image by $\rho$ of the arithmetic Frobenius lift $\sigma$ in $D_{\mathrm{v}}$ to have two eigenvalues with ratio $|\mathbb{F}_{\mathrm{v}}|$, because $\rho_{\mathrm{WD}}(\sigma)N = |\mathbb{F}_{\mathrm{v}}|N\rho_{\mathrm{WD}}(\sigma)$.

**Proposition 7.**

- $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$ *is reduced,* $\mathcal{O}$*-flat, Cohen-Macaulay of equi-dimension* 4 *over* $\mathcal{O}$*,*
- *Every irreducible component of* Spec $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}[1/p]$ *is formally smooth over* $L$*,*
- $R^{\square}_{\mathrm{v}}/(I^{\square,\zeta}_{\mathrm{v}}, \lambda)$ *is reduced,*
- *The generic point of every irreducible component of* $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$ *has characteristic zero.*
- *If* $\zeta$ *is distinct, then* $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$ *is geometrically irreducible of dimension 4 over* $\mathcal{O}$*.*
- *If* $\zeta$ *is trivial and* $L$ *is sufficiently large, every minimal prime of* $R^{\square}_{\mathrm{v}}/(I^{\square,\zeta}_{\mathrm{v}}, \lambda)$ *contains a unique minimal prime of* $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$*.*
- *If* $\zeta$ *is trivial, the arithmetic rank* $r(I^{\square,\zeta}_{\mathrm{v}})$*, in the sense of [11] for example, is* 2*.*

*A similar set of statements holds for* $R^{\square}_{\mathrm{v}}/I^{\square,\mathrm{St}}_{\mathrm{v}}$*.*

*Proof.* It follows from Exercise 18.13 in [36] and Proposition 5.8 (3) in [78] if $p > 2$ and Corollary B.10 in [77] if $p = 2$ that $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$ is Cohen-Macaulay. When $\zeta$ is trivial, Spec $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}_{\mathrm{v}}$ is a union of two types of universal rings, one for unramified liftings and the other for Steinberg liftings. The fibre Spec $R^{\square}_{\mathrm{v}}/(I^{\square,\zeta}_{\mathrm{v}}, \lambda)$ is reduced since it is covered by reduced schemes (because of Proposition 5.8 (3) in [78] if $p > 2$ and the proofs of Proposition B.8 and Proposition B.9 in [77] if $p = 2$; and because a localisation of a reduced scheme remains reduced).

To prove the last statement, suppose that $\zeta$ is trivial. Let $I_{\mathrm{v}}$ denote the inertia subgroup of $D_{\mathrm{v}}$ at v and $K_{\mathrm{v}}$ denote the kernel of the projection of $I_{\mathrm{v}}$ onto its maximal pro-$p$ quotient (of rank 1). The short exact sequence $1 \to I_{\mathrm{v}}/K_{\mathrm{v}} \to D_{\mathrm{v}}/K_{\mathrm{v}} \to D_{\mathrm{v}}/I_{\mathrm{v}} \to 1$ splits, and let $\tau$ (resp. $\sigma$) denote a topological generator of $I_{\mathrm{v}}/K_{\mathrm{v}} \simeq \mathbb{Z}_p$ (resp. $D_{\mathrm{v}}/I_{\mathrm{v}} \simeq \hat{\mathbb{Z}}$). Since $\overline{\rho}_{\mathrm{v}}$ is trivial when restricted to $K_{\mathrm{v}}$, so is any lifting $\rho : D_{\mathrm{v}} \to \mathrm{GL}_2(R)$ of $\overline{\rho}$ over an object $R$ in $\mathscr{C}$; and $\rho$ is determined by the images in $\mathrm{GL}_2(R)$ of $\tau$ and $\sigma$, subject to the condition that $\sigma\tau\sigma^{-1} = \tau^{|\mathbb{F}_{\mathrm{v}}|}$. It therefore follows that $R^{\square}_{\mathrm{v}}/I^{\square,\zeta}$ is given by the quotient of a power series ring in $4+4$ variables with coefficients in $\mathcal{O}$ by the radical $\sqrt{I}$ of an ideal $I$ generated by $2 + 4$ relations (2 because of the characteristic polynomial of $\rho(\tau)$ and 4 because of $\sigma\tau\sigma^{-1} = \tau^{|\mathbb{F}_{\mathrm{v}}|}$). The ideal $I$ is the intersection of two ideals– one corresponding to the 'unramified' liftings $\rho$ with trivial $\rho(\sigma)$ and the other $I^{\square,\zeta,\mathrm{St}}$ corresponding to the 'Steinberg' liftings $\rho$ with $\rho(\sigma)$ satisfying $|\mathbb{F}_{\mathrm{v}}|^2(\mathrm{tr}\,\rho(\sigma))^2 = (|\mathbb{F}_{\mathrm{v}}| + 1)^2 \det \rho(\sigma)$. Since $\sqrt{I}$ is the intersection of their radicals, $\sqrt{I}$ defines $I^{\square,\zeta}_{\mathrm{v}}$ and $r(I^{\square,\zeta}) = r(I) = 8 - 6 = 2$. □

Let $S_R$ (resp. $S_L$) denote the set of places v as in Proposition 7 with its corresponding deformation condition defined by the ideal $I^{\square,\zeta}_{\mathrm{v}} \subset R^{\square}_{\mathrm{v}}$ (resp. $I^{\square,\mathrm{St}}_{\mathrm{v}} \subset R^{\square}_{\mathrm{v}}$). As in [83], we will use the distinction between the cases– when $\zeta$ is trivial and when $\zeta$ is distinct– to 'avoid Ihara's lemma'.

## 2.6 $S_A$

**Lemma 8.** *We may and will find a finite place* v *of* $F$ *such that*

- v *does not divide* $p$*,*

- *if* $p > 2$, $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v}$ *is* not *congruent to* 1 *mod* $p$, $\overline{\rho}_{\mathrm{v}}$ *is unramified and* $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ *is has equal eigenvalues,*

- *if* $p = 2$, $\overline{\rho}_{\mathrm{v}}$ *is unramified and* $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ *has distinct eigenvalues.*

*For a such* v, $R_{\mathrm{v}}^{\square}$ *(i.e.* $/I_{\mathrm{v}}^{\square} = \varnothing$*) parameterises the twists of unramified liftings of* $\overline{\rho}_{\mathrm{v}}$ *and it reduced and Cohen-Macaylay. Furthermore,* $R_{\mathrm{v}}^{\square}/(\lambda, I_{\mathrm{v}}^{\square})$ *is reduced.*

*Proof.* When $p > 2$, see Proposition 5.5 (1) and Proposition 5.6 in (1) in [78]. When $p = 2$, see the proof of Lemma 0.4 in [77]. Indeed, the assumptions on $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ force any lifting of $\overline{\rho}_{\mathrm{v}}$ is unramified (or equivalently, not of Steinberg type) up to twist. □

We let $S_A$ be a set of such primes v.

## 2.7 $A_{\Sigma}^{\square}$ and $B_{\Sigma}^{\square}$

For every v in $T$, we let $A_{\mathrm{v}}^{\square}$ denote the quotient of $R_{\mathrm{v}}$ as defined above. In particular, we let

$$A_{\Sigma_p}^{\square} = (\hat{\bigotimes}_{\mathrm{v}\in S_p} A_{\mathrm{v}}^{\square})\hat{\otimes}_{\mathcal{O}[[\Delta_p\times\Delta_p]]}\Lambda.$$

Let

$$A_{\Sigma}^{\square} = A_{\Sigma_p}^{\square}\hat{\otimes}\hat{\bigotimes}_{\mathrm{v}\in T-S_p} A_{\mathrm{v}}^{\square}$$

and

$$B_{\Sigma}^{\square} = A_{\Sigma_p}^{\square}\hat{\otimes}\hat{\bigotimes}_{\mathrm{v}\in T-(S_p\cup S_R)} A_{\mathrm{v}}^{\square}.$$

It follows from sections above that the set of minimal primes of $B_{\Sigma}^{\square}$ is in bijection with the set of minimal primes of $\Lambda$. This plays a role in computing the connectedness dimension of $R_{\Sigma}$ in Proposition 9.

## 2.8 $\overline{\rho}$

**Definition**. For any object $R$ in $\mathcal{O}$, a continuous irreducible representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$ is said to be dihedral if it doe snot factor through any abelian quotient, but its restriction to $\mathrm{Gal}(\overline{F}/E)$ does for some quadratic extension $E$ over $F$. If this is the case, we say that it is $E$-dihedral.

Let

$$S = S_p \cup S_R \cup S_L \cup S_A \cup S_\infty$$

and suppose that $|S_L \cup S_\infty|$ is even. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$

- is unramified outside $S$,

- is trivial at every place in $S_p \cup S_R \cup S_L$

- is unramified at every place v in $S_A$; if $p > 2$, we assume that $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v}$ is not congruent to 1 mod $p$ and $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ has equal eigenvalues, while if $p = 2$, we assume that $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ has distinct eigenvalues.

## 2.9 Connected dimension

**Proposition 9.** *Suppose that $\zeta$ is trivial. For the connected dimension $c(R_\Sigma)$ of $R_\Sigma$ in the sense of [11] or [12],*

$$c(R_\Sigma) \geq [F:\mathbb{Q}] + \gamma_F - 2|S_R| - 1$$

*holds if $\overline{\rho}$ is not exceptional, while only*

$$c(R_\Sigma) \geq [F:\mathbb{Q}] + \gamma_F - 2|S_R| - 2$$

*holds if $\overline{\rho}$ is exceptional.*

*Proof.* Since $R_\Sigma^\square$ is a quotient of a power series in $\dim_\mathbb{F} H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho})$ variables over $A_\Sigma^\square$ with $\dim_\mathbb{F} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1))$ relations, it follows from Corollary 19.2.1 in [11] that

$$\begin{aligned} c(R_\Sigma^\square) &\geq c(A_\Sigma^\square) + \dim_\mathbb{F} H^1_\Sigma(F, \mathrm{ad}\,\overline{\rho}) - \dim_\mathbb{F} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 1 \\ &= c(A_\Sigma^\square) + \dim_\mathbb{F} H^0_\Sigma(F_S, \mathrm{ad}\,\overline{\rho}) - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - [F:\mathbb{Q}] - 1 \end{aligned}$$

by Proposition 5, where $\dim_\mathbb{F} H^0_\Sigma(F_S, \mathrm{ad}\,\overline{\rho}) = 0$ and $\dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) = 1$ (resp. 2) if $\overline{\rho}$ is not exceptional (resp. exceptional).

For a place v in $S_R$, $R_v^\square / I_v^{\square,\zeta}$ (where $\zeta$ is trivial) admits a presentation as the quotient of a power series over $\mathcal{O}$ with 8 variables by by 6 relations (Section 3 in [83]). It therefore follows that $A_\Sigma^\square$ admits a presentation as the quotient of a power series over $B_\Sigma^\square$ with $8|S_R|$ variables by $6|S_R|$ relations. By Proposition 1.8 in [89],

$$c(A_\Sigma^\square) \geq c(B_\Sigma^\square) + (8|S_R| - 6|S_R|) - 1.$$

On the other hand, since the set of minimal primes in $B_\Sigma$ and the set of minimal primes in $\Lambda$ are in bijection, one concludes as in the proof of Lemma 3.21 in [89] that

$$c(B_\Sigma^\square) \geq \dim B_\Sigma^\square/\lambda = (\dim A_\Sigma^\square - 4|S_R|) - 1.$$

Combining,

$$c(A_\Sigma^\square) \geq \dim A_\Sigma^\square - 2|S_R| - 2 = 1 + (1 + 2[F:\mathbb{Q}] + \gamma_F) + 4|T| - 2|S_R| - 2.$$

It therefore follows that

$$\begin{aligned} c(R_\Sigma^\square) &\geq c(A_\Sigma^\square) - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - [F:\mathbb{Q}] - 1 \\ &\geq ([2[F:\mathbb{Q}] + \gamma_F + 4|T| - 2|S_R|) - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - [F:\mathbb{Q}] - 1 \\ &= [F:\mathbb{Q}] + \gamma_F + 4|T| - 2|S_R| - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - 1 \end{aligned}$$

and therefore

$$\begin{aligned} c(R_\Sigma) &\geq [F:\mathbb{Q}] + \gamma_F + 4|T| - 2|S_R| - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) - 1 - (4|T| - 1) \\ &= [F:\mathbb{Q}] + \gamma_F - 2|S_R| - \dim_\mathbb{F} H^0(F_S, \mathrm{ad}\,\overline{\rho}(1)) \end{aligned}$$

□

## 3 Modular forms and Hecke operators

Let $D$ denote the quaternion algbera over $F$ ramified exactly at $S_L \cup S_\infty$ (where $|S_L \cup S_\infty|$ is assumed to be even). Let $G$ denote the algebraic group over $F$ defined by $D^\times$. We fix a maximal order $R$ of $D$ once for all and, for every finite place v not in $S_L$, we fix an isomorphism $(R \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v})^\times \simeq \mathrm{GL}_2(\mathcal{O}_{F_v})$.

Let $H = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$ and $H_v = \mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p)$ for every place v above $p$.

For every finite place v of $F$, the preimage by

$$\mathrm{GL}_2(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_2(\mathcal{O}_{F_v}/\mathrm{v}) = \mathrm{GL}_2(\mathbb{F}_v)$$

of the subgroup of upper triangular (resp. unipotent) matrices $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$) in $\mathrm{GL}_2(\mathbb{F}_v)$ will be denoted $B_v$ (resp. $B_v^+$).

Fix a set $\zeta$ of characters $\left\{ \zeta_v = \begin{pmatrix} \zeta_{v,1} & \\ & \zeta_{v,2} \end{pmatrix} : B_v/B_v^+ \to \mathcal{O}^\times \right\}_{v \in S_R}$ and a pair of tuples $(k, \ell) \in \mathbb{Z}^H \times \mathbb{Z}^H$. Define $V_{(k,\ell)}^\zeta$ to be

$$V_{(k,\ell)}^\zeta = \left( \bigotimes_{v \in S_p} \bigotimes_{\tau \in H_v} V_{(k_\tau, \ell_\tau)} \right) \otimes_{\mathcal{O}} \left( \bigotimes_{v \in S_R} \mathcal{O}(\zeta_v) \right)$$

where $V_{(k_\tau,\ell_\tau)}$ denote the representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ on a finite free $\mathcal{O}$-module, with action of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ defined in terms of the induction of the representation of the upper triangular Borel subgroup of weight $(\ell_\tau, k_\tau - 2 + \ell_\tau)$.

For an $\mathcal{O}$-algebra $R$, let $S_{(k,\ell)}^\zeta(R)$ denote the space of functions:

$$f : G(F) \backslash G(\mathbb{A}_F^\infty) \to V_{(k,\ell)}^\zeta \otimes_{\mathcal{O}} R.$$

If $U$ is a subgroup of $G(\mathbb{A}^\infty)$ such that $U_v \subseteq G(\mathcal{O}_{F_v})$ for every v in $S_p$ and $U_v \subseteq B_v$ for v in $S_R$, we let $U$ acts on $S_{(k,\ell)}^\zeta(R)$ by

$$(\gamma f)(g) = (\gamma_{p \cup R}) f(g\gamma)$$

where $\gamma_{p \cup R}$ is the projection of $\gamma$ into the $S_p \cup S_R$-component.

Let $S_{(k,\ell)}^\zeta(U, R)$ denote the subset of functions $f$ in $S_{(k,\ell)}^\zeta(R)$ such that $\gamma f = f$ for every $\gamma$ in $U$. When $\zeta$ is trivial, we simply write $S_{(k,\ell)}(U, R)$.

Let $U_{\Sigma_{Q,N}}^{[r],+}$ (resp. $U_{\Sigma_{Q,N}}^{[r]}$) the open subgroup $U$ of $G(\mathbb{A}^\infty)$ defined in terms of the deformation data $\Sigma_{Q,N}$ such that

- $U_v$, for v in $S_p$, is the subgroup of matrices in $G(\mathcal{O}_{F_v}) = \mathrm{GL}_2(\mathcal{O}_{F_v})$ which reduce mod the $r$-th power of v to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$.
- $U_v = G(\mathcal{O}_{F_v})$ for v in $S_L$,

- $U_v$ is the pre-image, by $\mathrm{GL}_2(\mathscr{O}_{F_v}) \to \mathrm{GL}_2(\mathbb{F}_v)$, of the subgroup of matrices in $\mathrm{GL}_2(\mathbb{F})$ of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$) for v in $S_R$ (resp. $S_A$).

- $U_v$, for v in $S_{Q,N}$, is the subgroup of matrices in $G(\mathscr{O}_{F_v}) = \mathrm{GL}_2(\mathscr{O}_{F_v})$ which reduces mod v to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (resp. to $\begin{pmatrix} * & * \\ 0 & \ker\left(\mathbb{F}_v^\times \twoheadrightarrow \Delta_{Q,v}\right) \end{pmatrix}$ where $\Delta_{Q,v}$ is the maximal pro-$p$ quotient of $\mathbb{F}_v^\times = (\mathscr{O}_F/v)^\times$).

By definition $U^{[r]}_{\Sigma_{Q,N}}$ is sufficiently small in the sense of Section 2.4 in [75] (see [75] and Lemma 3.2 in [70] when $p = 2$) and

$$U^{[r],+}_{\Sigma_{Q,N}}/U^{[r]}_{\Sigma_{Q,N}} \simeq \prod_{v \in S_{Q,N}} \Delta_{Q,v}.$$

For v not in $S$, we let $T_v$ (resp. $S_v$) denote the Hecke operator acting on $S^{\zeta}_{(k,\ell)}(U, R)$ corresponding to $\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix}$). The module $S^{\zeta}_{(k,\ell)}(U, R)$ also comes equipped with action of

- $U_v$ (resp. $S_v$) for every place v above $p$ corresponding to $\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix}$) normalised by multiplying $\prod_{\tau \in H_v} \tau(\pi_v)^{-\ell_\tau}$ (resp. $\prod_{\tau \in H_v} \tau(\pi_v)^{-(k_\tau + 2\ell_\tau - 2)}$),

- for every place v above $p$, an element $t_v = \begin{pmatrix} t_{v,1} & \\ & t_{v,2} \end{pmatrix}$ in the diagonal torus $\begin{pmatrix} \mathscr{O}^\times_{F_v} & 0 \\ 0 & \mathscr{O}^\times_{F_v} \end{pmatrix}$ naturally acts (without being normalised) and we follow Definition 2.23 in [42] to define

$$\langle t \rangle = \prod_v t^{-1}_{v,2} t_v = \prod_v \begin{pmatrix} t_{v,1}/t_{v,2} & \\ & 1 \end{pmatrix}$$

for $t = (t_v)$ in $\prod_v \begin{pmatrix} \mathscr{O}^\times_{F_v} & 0 \\ 0 & \mathscr{O}^\times_{F_v} \end{pmatrix}$.

Let $T_{(k,\ell),\Sigma_{Q,N}}(U^{[r]}_{\Sigma_{Q,N}}, R)$ denote the Hecke algebra generated by the images in $\mathrm{End}_R(S^{\zeta}_{(k,\ell)}(U^{[r]}_{\Sigma_{Q,N}}, R))$ of $T_v$ and $S_v$ for v not in $S \cup S_{Q,N}$, $U_v$ for v in $S_p$ and the $S_\tau$. When $R = \mathscr{O}$, we shall not make reference to $R$. When $k = 2$ and $\ell = 0$, write 2 in place of $(k, \ell)$.

For $R = \mathscr{O}$ or $L/\mathscr{O}$, Section 2.4 in [42] defines the Hida idempotent $e$ on

$$S^{\zeta}_{(k,\ell)}(U^{[r]}_{\Sigma_{Q,N}}, R),$$

and

$$T_{(k,\ell),\Sigma_{Q,N}}(U^{[r]}_{\Sigma_{Q,N}}, R),$$

and we let

$$eS^{\zeta}(U_{\Sigma_{Q,N}}, R) = \lim_{r \to} S^{\zeta}_{(k,\ell)}(U^{[r]}_{\Sigma_{Q,N}}, R)$$

and

$$eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}}, R) = \lim_{\leftarrow r} eT_{2,\Sigma_{Q,N}}(U^{[r]}_{\Sigma_{Q,N}}, R).$$

When $S_{Q,N} = \varnothing$, we simply write $eS^{\zeta}(U_{\Sigma}, R)$ and $eT_{\Sigma}(U_{\Sigma}, R)$ respectively. If $R = \mathscr{O}$, we make omit our reference to the coefficient $R$. Naturally, $eS^{\zeta}(U_{\Sigma_{Q,N}})$ and $eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ are $\Lambda$-algebras via $\langle\ \rangle$.

When $\zeta$ is trivial and it is necessary for us to emphasise it, we omit the reference to $\zeta$ in the notation.

Given a maximal ideal of $eT_{\Sigma}(U_{\Sigma})$ for the deformation data $\Sigma$ in which the set of characters $\zeta$ is trivial, the congruence with $eS(U_{\Sigma})/\lambda$ allows us to define the corresponding maximal ideal $\mathfrak{m} \subset eT_{\Sigma}(U_{\Sigma})$ for *any* $\Sigma$ as above (since $\zeta$ is trivial mod $\lambda$). Let $\mathfrak{m}_{Q,N}$ denote the pre-image of $\mathfrak{m}$ by

$$eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}}) \twoheadrightarrow eT_{\Sigma}(U_{\Sigma}).$$

**Lemma 10.**
- *$eS^{\zeta}(U_{\Sigma_{Q,N}}, L/\mathscr{O})^{\vee}_{\mathfrak{m}_{Q,N}}$ is free over $\Lambda$ of rank* $\dim_{\mathbb{F}} eS_2(U_{\Sigma}^{[r]}, \mathbb{F})$ *for sufficiently large $r$.*
- *$eS^{\zeta}(U_{\Sigma_{Q,N}}, L/\mathscr{O})^{\vee}_{\mathfrak{m}_{Q,N}}/\mathfrak{m}_{\Lambda}$, where $\mathfrak{m}_{\Lambda}$ is the maximal ideal of $\Lambda$, is isomorphic to $eS^{\zeta}(U^{[1]}_{\Sigma_{Q,N}}, \mathbb{F})^{\vee}_{\mathfrak{m}_{Q,N}}$.*

*Proof.* See Proposition 2.20 in [42] for the first assertion– essentially, it follows from $U_{\Sigma_{Q,N}} \subset U_{\Sigma}$ being sufficiently small. The second assertion follows by definition. ☐

Suppose that $\overline{\rho}$ is modular, in the sense that $\overline{\rho} \simeq \overline{\rho}_{\mathfrak{m}}$ for a non-Eisenstein maximal ideal $\mathfrak{m}$ in $T_{\Sigma}(U_{\Sigma})$. There exists a continuous representation

$$\rho_{\Sigma_{Q,N}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}})$$

lifting $\overline{\rho}$, which is unramified outside $S \cup S_{Q,N}$ and which satisfies

$$\mathrm{tr}\,\rho_{\Sigma_{Q,N}}(\mathrm{Frob}_{\mathrm{v}}) = T_{\mathrm{v}}$$

and

$$\det \rho_{\Sigma_{Q,N}}(\mathrm{Frob}_{\mathrm{v}}) = (\mathbf{N}_{F/\mathbb{Q}}\mathrm{v})S_{\mathrm{v}}$$

for every v not lying in $S \cup S_{Q,N}$ (where $\mathrm{Frob}_{\mathrm{v}}$ is a geometric Frobenius lift).

For every v in $S_{Q,N}$, the restriction of $\rho_{\Sigma_{Q,N}}$ at v is a lifting of a direct sum of unramified characters $\overline{\chi}_{\mathrm{v},1}, \overline{\chi}_{\mathrm{v},2} : D_{\mathrm{v}} \to T^{\times}_{\Sigma_{Q,N}}$. Let $\alpha_{\mathrm{v}}$ (resp. $\beta_{\mathrm{v}}$) be a root in $eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}$ of the characteristic polynomial of $\rho_{\Sigma_{Q,N}}(\mathrm{Frob}_{\mathrm{v}})$, lifting $\overline{\chi}_{\mathrm{v},1}(\mathrm{Frob}_{\mathrm{v}})$ (resp. $\overline{\chi}_{\mathrm{v},2}(\mathrm{Frob}_{\mathrm{v}})$) in $\mathbb{F}$; this is given by Hensel's lemma. We define a 'Hecke' operator

$$U_{\pi} : eS^{\zeta}(U^{[r]}_{\Sigma_{Q,N}}, L/\mathscr{O})_{\mathfrak{m}_{Q,N}} \to eS^{\zeta}(U^{[r]}_{\Sigma_{Q,N}}, L/\mathscr{O})_{\mathfrak{m}_{Q,N}}$$

corresponding to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ where $\pi$ is an element of $F_{\mathrm{v}}$ with non-negative valuation.

Define the quotient $H^{\zeta}_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ of $eS^{\zeta}(U_{\Sigma_{Q,N}}, L/\mathscr{O})^{\vee}_{\mathfrak{m}_{Q,N}}$ to be the Pontrjagin of the submodule

$$H^{\zeta}_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})^{\vee} = \left(\prod_{\mathrm{v}\in S_{Q,N}} (U_{\pi_{\mathrm{v}}} - \beta_{\mathrm{v}})\right) eS^{\zeta}(U_{\Sigma_{Q,N}}, L/\mathscr{O})_{\mathfrak{m}_{Q,N}} \subset eS^{\zeta}(U_{\Sigma_{Q,N}}, L/\mathscr{O})_{\mathfrak{m}_{Q,N}}$$

($\pi_v$ is a uniformiser at v) cut out by the deformation data $\Sigma_{Q,N}$ and the local Langlands correspondence (see Section 4.2 in [42]). Analogously, one can define the quotient $eS^{\zeta}(U^{+}_{\Sigma_{Q,N}}, L/\mathcal{O})^{\vee}_{\mathfrak{m}_{Q,N}} \twoheadrightarrow H^{\zeta}_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}})$ with $U^{[r],+}_{\Sigma_{Q,N}}$ in place of $U^{[r]}_{\Sigma_{Q,N}}$.

We may define the quotient $H^{\zeta}_{\Sigma_{Q,N}}(U^{[r]}_{\Sigma_{Q,N}}, \mathcal{O}/\lambda^s)$ of $eS^{\zeta}(U_{\Sigma_{Q,N}}, \mathcal{O}/\lambda^s)_{\mathfrak{m}_{Q,N}}$ similarly; $H^{\chi}_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}}, \mathcal{O})$ is the inverse limit of $H^{\zeta}_{\Sigma_{Q,N}}(U^{[r]}_{\Sigma_{Q,N}}, \mathcal{O}/\lambda^s)$.

There exists a character

$$\chi_v : F_v^{\times} \to \left(eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}\right)^{\times}$$

such that, for every $\pi$ in $F_v$ with non-negative valuation with respect to v, the operator $U_{\pi}$ acts on $H_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}}, \mathcal{O})$ by $\chi_v(\pi)$; and the restriction of $\rho_{\Sigma_{Q,N}}$ at v in $S_{Q,N}$ is of the form $(\chi_v \circ \mathrm{Art}^{-1}_{F_v}) \oplus \chi_{v,2}$ where $\chi_{v,2}$ is an unramified lifting of $\overline{\chi}_{v,2}$.

Let

$$T_{\Sigma_{Q,N}} \subset \mathrm{End}(H^{\zeta}_{\Sigma_{Q,N}})$$

denote the image of $T_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}$ in $\mathrm{End}(H^{\zeta}_{\Sigma_{Q,N}})$. When $S_{Q,N} = \varnothing$, we simply write $T_{\Sigma}$ and $H^{\zeta}_{\Sigma}$ for $T_{\Sigma_{Q,N}}$ and $H^{\zeta}_{\Sigma_{Q,N}}$.

Let

$$H^{\square,\zeta}_{\Sigma_{Q,N}} = H^{\zeta}_{\Sigma_{Q,N}} \otimes_{R_{\Sigma_{Q,N}}} R^{\square}_{\Sigma_{Q,N}}$$

for which we write $H^{\square,\zeta}_{\Sigma}$ when $S_{Q,N} = \varnothing$.

By definition, $H^{\zeta}_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}})$ comes equipped with action of $U^{[r],+}_{\Sigma_{Q,N}}/U^{[r]}_{\Sigma_{Q,N}} \simeq \Delta_{Q,N} := \prod_{v \in S_{Q,N}} \Delta_{Q,v}$.

**Proposition 11.**

- *$H^{\zeta}_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ is a finite free (hence flat) module over $\Lambda[\Delta_{Q,N}]$, and the coinvariants of $H^{\zeta}_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ by $\mathcal{O}[\Delta_{Q,N}]$ are $H^{\zeta}_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}})$.*

- *The map $\left(\prod_{v \in S_{Q,N}}(U_{\pi_v} - \beta_v)\right)^{\vee} : H^{\zeta}_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}}, \mathcal{O}) \to H^{\zeta}_{\Sigma}(U_{\Sigma}, \mathcal{O})$, which sends $\phi \in H^{\zeta}_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}}, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(H_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})^{\vee}, L/\mathcal{O})$ to the continuous homomorphism*

$$\left[eS^{\zeta}(U_{\Sigma}, L/\mathcal{O})_{\mathfrak{m}} \hookrightarrow eS^{\zeta}(U^{+}_{\Sigma_{Q,N}}, L/\mathcal{O})_{\mathfrak{m}_{Q,N}} \xrightarrow{\prod_v (U_{\pi_v} - \chi_{v,1}(\mathrm{Frob}_v))} H_{\Sigma_{Q,N}}(U^{+}_{\Sigma_{Q,N}})^{\vee} \xrightarrow{\phi} L/\mathcal{O}\right] \in H^{\zeta}_{\Sigma}(U_{\Sigma}, \mathcal{O}),$$

  *is an isomorphism.*

*Proof.* See Lemma 4.9 in [42]. For the second assertion, see also Lemma 3.2.2 in [22]. □

# 4 $\overline{\rho}$ is irreducible with insoluble image

Suppose that $p > 2$.

**Lemma 12.** *Suppose that $\overline{\rho}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$. If $p = 5$ and the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, we furthermore assume that the kernel of the projective representation of $\overline{\rho}$ does not fix $F(\zeta_5)$. For every $N \geq 1$, there exists a finite set $S_{Q,N}$ of finite places v of $F$ such that*

- $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v} \equiv 1 \; mod \; p^N$,
- $\overline{\rho}_{\mathrm{v}}$ *is a direct sum of distinct unramified characters,*
- $|S_{Q,N}| = q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1))$, *i.e.* $|S_{Q,N}|$ *is independent of* $N$,
- *if we let* $\Sigma_{Q,N}$ *denote the deformation data*

$$(S \cup S_{Q,N}, T, \ldots, ),$$

*then* $R^{\square}_{\Sigma_{Q,N}}$ *is topologically generated over* $A^{\square}_{\Sigma}$ *by* $r = q - [F : \mathbb{Q}] - 1$ *elements.*

*Proof.* This is standard and follows from Proposition 5. See [75]. $\square$

Suppose that $p = 2$. Let $F_N = F(\zeta_{2^N})$ and $K$ be the subfield of $\overline{F}$ fixed by $\ker \mathrm{ad}\,\overline{\rho}$. Let $K_N$ denote the compositum of $K$ and $F_N$.

**Definition**. Let $N_{\mathrm{kw}}$ be the largest integer amongst those $N \geq 2$ such that the totally real subfield of $F_N$ is contained in $F$.

We have an inflation-restriction exact sequence:

$$\to H^0(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho}(1)/\mathbb{F}) \to H^1(\mathrm{Gal}(F_N/F), \mathbb{F}) \to H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho}(1)) \to$$

and if $\overline{\rho}$ is irreducible with insoluble image, $H^0(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho}(1)/\mathbb{F}) = 0$ (Lemma 4.3 in [56]).

Recall, by definition, $H^1_{\Sigma^\perp}(F_S, \mathrm{ad}\,\overline{\rho}(1))$ to be the kernel of

$$H^1(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}(1)) \to \bigoplus_{\mathrm{v}} H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}(1))/L_{\mathrm{v}}^{\perp}$$

where the direct sum ranges over the union of $S_\infty$ and $S_{Q,N}$; at every infinite place v of $F$, we have

$$L_{\mathrm{v}}^{\perp} = \left(H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}(1))/L_{\mathrm{v}}\right)^{\vee} \subset H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}(1))^{\vee} = H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}(1)),$$

with

$$\dim_{\mathbb{F}} L_{\mathrm{v}}^{\perp} = \begin{cases} 0 & \text{if } \overline{\rho}_{\mathrm{v}} \text{ is non-trivial,} \\ 1 & \text{if } \overline{\rho}_{\mathrm{v}} \text{ is trivial,} \end{cases}$$

while at v in $S_{Q,N}$, we have $L_{\mathrm{v}}^{\perp} = H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})^{\perp}$. Parenthetically, it follows from the local Euler characteristic formula and the local duality, one sees $\dim H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho}(1)) = \dim H^1(D_{\mathrm{v}}, \mathrm{ad}\,\overline{\rho})$ is 0 (resp. 4) if $\overline{\rho}_{\mathrm{v}}$ is non-trivial (resp. trivial).

**Definition**. As in Section 2, we define $H^1_{\Sigma_{Q,N}}(F_S, \mathbb{F})$ to be the subspace of $H^1(\mathrm{Gal}(F_S/F), \mathbb{F})$ defined by local conditions $L_{\mathrm{v}} = H^1(D_{\mathrm{v}}, \mathbb{F})$ at $S_\infty$ and $S_{Q,N}$ (whether v is finite or not). It follows from local Tate duality that, for infinite v, we have

$$\dim_{\mathbb{F}} L_{\mathrm{v}} = \begin{cases} 0 & \text{if } \overline{\rho}_{\mathrm{v}} \text{ is non-trivial,} \\ 1 & \text{if } \overline{\rho}_{\mathrm{v}} \text{ is trivial,} \end{cases}$$

As in Section 2, we define the dual Selmer group $H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathbb{F})$ as the kernel of

$$H^1(\mathrm{Gal}(F_S/F),\mathbb{F}) \to \bigoplus_{\mathrm{v}} H^1(D_{\mathrm{v}},\mathbb{F})/L_{\mathrm{v}}^\perp$$

where v ranges over $S_\infty$ and $S_{Q,N}$. The dimension $\dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F_S,\mathbb{F})$ is computed as

$$\begin{aligned} & \dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathbb{F}) + H^0(F_S,\mathbb{F}) - H^0(F_S,\mathbb{F}(1)) \\ + \;& \textstyle\sum_{\mathrm{v}\in S_\infty} \dim_{\mathbb{F}} L_{\mathrm{v}} - \left[\dim_{\mathbb{F}} H^0(D_{\mathrm{v}},\mathbb{F}) - \mathbf{N}_{\overline{F}_{\mathrm{v}}/F_{\mathrm{v}}} H^0(D_{\mathrm{v}},\mathbb{F})\right] + \sum_{\mathrm{v}\in S_{Q,N}} H^1(D_{\mathrm{v}},\mathbb{F}) - H^0(D_{\mathrm{v}},\mathbb{F}) \\ = \;& \dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathbb{F}) + |S_{Q,N}|. \end{aligned}$$

By definition, $H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathbb{F})$ maps to $\dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathrm{ad}\,\overline{\rho}(1))$, and they are isomorphic if $\overline{\rho}$ is irreducible with insoluble image (as observed above); in particular, under the assumption,

$$\dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathbb{F}) = \dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F_S,\mathrm{ad}\,\overline{\rho}(1)).$$

**Lemma 13.** *Suppose that $\overline{\rho}$ has insoluble image. For every $N \geq N_{\mathrm{kw}}$, there exists a finite set $S_{Q,N}$ of finite places* v *of $F$ such that*

- $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v} \equiv 1$ *mod* $p^N$,
- $\overline{\rho}_{\mathrm{v}}$ *is a direct sum of distinct unramified characters,*
- $|S_{Q,N}| = q$, *where* $q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\ \overline{\rho}(1)) - 2$,
- *if we let $\Sigma_{Q,N}$ denote the deformation data*

$$(S \cup S_{Q,N}, T, \ldots, ),$$

*then $R^\square_{\Sigma_{Q,N}}$ is topologically generated over $A^\square_\Sigma$ by $r = 2q - [F:\mathbb{Q}] + 1$ elements.*

*Proof.* Because $p = 2$, the adjoint representation $\mathrm{ad}\,\overline{\rho}$ is self-dual, i.e., $\mathrm{ad}\,\overline{\rho}(1) \simeq \mathrm{ad}\,\overline{\rho}$. Since $\overline{\rho}$ has insoluble image,

$$H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\ \overline{\rho}(1)) = 0$$

(see Lemma 4.3 in [56] for example).

Suppose that $\phi$ is a class in $H^1(\mathrm{Gal}(F_S/F),\mathrm{ad}\,\overline{\rho}(1))$ which has a *non-trivial* restriction in $H^1(\mathrm{Gal}(F_S/F_N),\mathrm{ad}\,\overline{\rho}(1))$, and therefore in $H^1(\mathrm{Gal}(F_S/K_N),\mathrm{ad}\,\overline{\rho}(1))$. In this case, we have

$$0 \to H^1(\mathrm{Gal}(K_N/F_N),\mathrm{ad}\,\overline{\rho}(1)) \to H^1(\mathrm{Gal}(F_S/F_N),\mathrm{ad}\,\overline{\rho}(1)) \to H^1(\mathrm{Gal}(F_S/K_N),\mathrm{ad}\,\overline{\rho}(1)),$$

where $H^1(\mathrm{Gal}(K_N/F_N),\mathrm{ad}\,\overline{\rho}(1)) = 0$ and $H^1(\mathrm{Gal}(F_S/K_N),\mathrm{ad}\,\overline{\rho}(1)) = \mathrm{Hom}(\mathrm{Gal}(F_S/K_N),\mathrm{ad}\,\overline{\rho}(1))$.

In particular, $\phi(\mathrm{Gal}(F_S/K_N))$ is a non-trivial $\mathrm{Gal}(K_N/F_N)$-submodule of $\mathrm{ad}\,\overline{\rho}(1)$. One may then find an element $\gamma$ of $\mathrm{Gal}(K_N/F_N)$, hence of $\mathrm{Gal}(F_S/F_N)$, satisfying the property that given a non-trivial irreducible $\mathrm{Gal}(K_N/F_N)$-module $Z$ (the scalars in $M_2(\mathbb{F})$) of $\mathrm{ad}\,\overline{\rho}$, the image $\mathrm{ad}\,\overline{\rho}(\gamma)$ of $\gamma$ has an eigenvalue other than 1 and has an eigenvalue 1 on $Z$. It then follows that, either $\gamma$ or its shift by an element of $\phi(\mathrm{Gal}(F_S/K_N))$ satisfies the condition that $\phi(\gamma)$ is *not* contained in $(\gamma-1)\mathrm{ad}\,\overline{\rho}$. By the Cebotarev density theorem, $\gamma$ gives rise to a finite place v of $F$ such that

- $\gamma$ equals $\mathrm{Frob}_v$ (up to conjugacy),
- $\mathbf{N}_{F/\mathbb{Q}} v \equiv 1 \bmod p^N$,
- v splits completely in $F_N$,
- $\overline{\rho}_v$ is the direct sum of character $\overline{\chi}_{v,1}$ and $\overline{\chi}_{v,2}$ and the restriction of $\phi$ at v lies non-trivially in

$$H^1(D_v/I_v, \mathrm{ad}\,\overline{\chi}_{v,1}(1)) = H^1(D_v/I_v, \mathrm{ad}\,\overline{\rho}(1))/L_v^\perp \simeq \mathbb{F}$$

We apply the argument repeatedly to an $\mathbb{F}$-basis of $H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) \subset H^1(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}(1))$ that restricts to a non-trivial class in $H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho}(1))$. The resulting subspace $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1))$ of $H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1))$ therefore lies in the kernel of $H^1(\mathrm{Gal}(F_S/F), \mathrm{ad}\,\overline{\rho}(1)) \to H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho}(1))$; and the latter is isomorphic to $H^1(F_N, \mathbb{F})$ by inflation-restriction, and $\dim_{\mathbb{F}} H^1(\mathrm{Gal}(F_N/F), \mathbb{F}) = 2$ for $N > N_{\mathrm{kw}}$, since the maximal elementary abelian 2-quotient of $\mathrm{Gal}(F_N/F)$ is of rank 2.

One can indeed establish that $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1))$ equals the kernel $H^1(\mathrm{Gal}(F_N/F), \mathbb{F})$; this is in stark contrast to the setting in [87] whose Proposition 2.21 observes $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1))$ is strictly contained in $H^1(\mathrm{Gal}(F_N/F), \mathbb{F})$ albeit under the assumption that there is at least one infinite place v at which $\overline{\rho}$ is non-trivial. To see the equality in our setting, we observe, for every infinite place v, the image of

$$H^1(\mathrm{Gal}(F_N/F), \mathbb{F}) \to H^1(D_v, \mathbb{F}) \xrightarrow{f} H^1(D_v, \mathrm{ad}\,\overline{\rho})$$

($F_N$ is no longer totally real) lies in $L_v^\perp$. It suffices to show that the image $\mathrm{Im}(f)$ of $f$ equals $L_v^\perp$. As $f$ is part of the exact sequence

$$\to H^1(D_v, \mathbb{F}) \xrightarrow{f} H^1(D_v, \mathrm{ad}\,\overline{\rho}) \xrightarrow{g} H^1(D_v, (\mathrm{ad}\,\overline{\rho})/\mathbb{F}) \to$$

with dual

$$\leftarrow H^1(D_v, \mathbb{F}) \xleftarrow{f^\vee} H^1(D_v, \mathrm{ad}\,\overline{\rho}) \xleftarrow{g^\vee} H^1(D_v, (\mathrm{ad}\,\overline{\rho})/\mathbb{F})^\vee = H^1(D_v, (\mathrm{ad}^0\overline{\rho})^\vee(1)) = H^1(D_v, \mathrm{ad}^0\overline{\rho}) \leftarrow$$

we see that $\mathrm{Im}(f)^\perp = \mathrm{coker}(f) = \ker(f^\vee) = \mathrm{Im}(g^\vee) = L_v$.

For v in $S_{Q,N}$,

$$\dim L_v - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = \dim H^1(D_v, \mathrm{ad}\,\overline{\rho}) - \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = \dim_{\mathbb{F}} H^0(D_v, \mathrm{ad}\,\overline{\rho}) = 2$$

It then follow from Proposition 5 that $\dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\overline{\rho})$ is computed by $\dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1)) - 1 + 2|S_{Q,N}| - [F:\mathbb{Q}] = 2 - 1 + 2q - [F:\mathbb{Q}] = 2q - [F:\mathbb{Q}] + 1$, where $|S_{Q,N}| = q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - \dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1)) = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 2$. $\square$

**Definition**. When $p = 2$, we let $\nabla_{Q,N}$ denote the group of characters of the Galois group of the maximal abelian pro-$p$ extension of $F$ un ramified outside $S_{Q,N}$ which are deformations/liftings of the trivial character over $\mathbb{F}$. This acts freely on $R^\square_{\Sigma_{Q,N}}$ 'by twisting'. As observed in Lemma 5.10 of [56], $\nabla_{Q,N}$ has rank $\dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F_S, \mathbb{F})$; if we let $\nabla_Q$ denote the $\mathscr{O}$-algebra of $\dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F_S, \mathbb{F})$-copies of $\mathbb{Z}_p$, then we have a surjection $\nabla_Q \to \nabla_{Q,N}$.

**Definition**. By slight abuse of notation, we let $R^{\square}_{\Sigma_{Q,N}}/\nabla_{Q,N}$ denote the subring of elements in $R^{\square}_{\Sigma_{Q,N}}$ which are invariant under action of $\nabla_{Q,N}$. By definition,

$$\dim R^{\square}_{\Sigma_{Q,N}}/\nabla_{Q,N} = \dim R^{\square}_{\Sigma_{Q,N}} - \dim_{\mathbb{F}} H^1_{\Sigma_{Q,N}}(F_S, \mathbb{F}) = \dim R^{\square}_{\Sigma_{Q,N}} - (2 + |S_{Q,N}|)$$

Following Lemma 12 and Lemma 13, we let $A^{\square}_{\Sigma_Q}$ denote the formal power series ring over $A^{\square}_{\Sigma}$ with $r$ variables, with the variable chosen such that $R^{\square}_{\Sigma_{Q,N}}$ is a quotient of $A^{\square}_{\Sigma_Q}$:

$$A^{\square}_{\Sigma_Q} \to R^{\square}_{\Sigma_{Q,N}}.$$

If $p = 2$, we furthermore let $A^{\square,\nabla}_{\Sigma_Q}$ denote the formal power series ring over $A^{\square}_{\Sigma}$ with $r-(2+q)$ variables, similarly defining a surjection

$$A^{\square,\nabla}_{\Sigma_Q} \to R^{\square}_{\Sigma_{Q,N}}/\nabla_{Q,N}.$$

Let $\Delta_Q$ be the free $\mathbb{Z}_p$-module $\mathbb{Z}^q$ of rank $q$. For every $N$, $\Delta_Q$ surjects onto

$$\Delta_{Q,N} = \prod_{\mathrm{v} \in S_{Q,N}} \Delta_{\mathrm{v}}.$$

Fix an isomorphism

$$\Lambda[[\Delta_Q]] \simeq \Lambda[[S_1, \ldots, S_q]].$$

Let $J$ denote the kernel of

$$\Lambda \hat{\otimes} R^{\square}[[\Delta_Q]] = \Lambda \hat{\otimes} R^{\square}[[S_1, \ldots, S_q]] \to \Lambda$$

which sends every variable in $\Delta_Q$ to 1 and all $4|T| - 1$ variables in $R^{\square}$ to 0.

**Lemma 14.** *Let $\triangle$ be a minimal ideal of $\Lambda$.*

- *If $\zeta$ is distinct (i.e. for every* v *in $S_R$, $\zeta_{\mathrm{v},1}$ and $\zeta_{\mathrm{v},2}$ are distinct), then* $\mathrm{Spf}\, A^{\square}_{\Sigma} \otimes \Lambda/\triangle$ *is $\mathscr{O}$-flat and geometrically irreducible of relative dimension over $\mathscr{O}$*

$$1 + 2[F : \mathbb{Q}] + \gamma_F + 4|T|.$$

- *If $\zeta$ is trivial (i.e. for every* v *in $S_R$, $\zeta_{\mathrm{v},1}$ and $\zeta_{\mathrm{v},2}$ both trivial) and if $L$ is sufficiently large, then* $\mathrm{Spf}\, A^{\square}_{\Sigma}\Lambda/\triangle$ *is equi-dimensional of relative dimension over $\mathscr{O}$*

$$1 + 2[F : \mathbb{Q}] + \gamma_F + 4|T|.$$

*Furthermore,*

- *every minimal prime of $A^{\square}_{\Sigma} \otimes \Lambda/(\triangle, \lambda)$ contains a unique prime of $A^{\square}_{\Sigma} \otimes \Lambda/\triangle$,*
- *$A^{\square}_{\Sigma}$ is $\mathscr{O}$-flat and Cohen-Macaulay,*
- *$A^{\square}_{\Sigma}/\lambda$ is generically reduced.*

*Proof.* This follows from Section 2. See Lemma 9 in [75]. □

Let $H^{\square}_{\Sigma_{Q,N}} = H_{\Sigma_{Q,N}} \otimes_{\mathcal{O}} R^{\square}$, and $T^{\square}_{\Sigma_{Q,N}} = T_{\Sigma_{Q,N}} \otimes_{\mathcal{O}} R^{\square}$ where $T_{\Sigma_{Q,N}}$ is defined as in Section 3. The Taylor-Wiles 'level $N$-modules'

$$\begin{array}{ccccccc} & & \Lambda\hat{\otimes}_{\mathcal{O}} R^{\square}[[\Delta_Q]] & & & & \\ & & \downarrow & & & & \\ A^{\square}_{\Sigma_Q} = A^{\square}_{\Sigma}[[X_1,\ldots,X_r]] & \longrightarrow & R^{\square}_{\Sigma_{Q,N}} & \longrightarrow & T^{\square}_{\Sigma_{Q,N}} & \subset & \mathrm{End}(H^{\square}_{\Sigma_{Q,N}}) \\ & & \downarrow & & \downarrow & & \\ & & R_{\Sigma_{Q,N}} & \longrightarrow & T_{\Sigma_{Q,N}} & & \\ & & \downarrow & & \downarrow & & \\ & & R_{\Sigma} & \longrightarrow & T_{\Sigma} & & \end{array}$$

if $p > 2$ and $R^{\square}_{\Sigma_{Q,N}}/\nabla_{Q,N}$ (resp. $A^{\square,\nabla}_{\Sigma_Q}$) in place of $R^{\square}_{\Sigma_{Q,N}}$ (resp. $A^{\square}_{\Sigma_Q}$) if $p = 2$; and they 'patch' together to define

$$\begin{array}{ccccccc} & & \Lambda\hat{\otimes}_{\mathcal{O}} R^{\square}[[\Delta_Q]] & & & & \\ & & \downarrow & & & & \\ A^{\square}_{\Sigma_Q} & \longrightarrow & R^{\square}_{\Sigma_Q} & \longrightarrow & T^{\square}_{\Sigma_Q} & \subset & \mathrm{End}(H^{\square}_{\Sigma_Q}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & R_{\Sigma} & \longrightarrow & T_{\Sigma} & \subset & \mathrm{End}(H_{\Sigma}) \end{array}$$

if $p > 2$ and

$$\begin{array}{ccccccc} & & \Lambda\hat{\otimes}_{\mathcal{O}} R^{\square}[[\Delta_Q]] & & & & \\ & & \downarrow & & & & \\ A^{\square}_{\Sigma_Q} & \longrightarrow & R^{\square}_{\Sigma_Q}/\nabla_Q & \longrightarrow & T^{\square}_{\Sigma_Q} & \subset & \mathrm{End}(H^{\square}_{\Sigma_Q}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & R_{\Sigma} & \longrightarrow & T_{\Sigma} & \subset & \mathrm{End}(H_{\Sigma}) \end{array}$$

**Remark**. When $p = 2$, action of $\nabla_Q$ is 'twist' by global characters, whilst action of $\Delta_Q$ manifests itself as the 'diamond' operator.

**Lemma 15.**

$$H^{\square}_{\Sigma_Q,\nabla}/J \simeq H_{\Sigma}.$$

*Proof.* Standard. □

**Theorem 16.** *$H^{\square}_{\Sigma_Q}$, with trivial $\zeta$, is a faithful module over $A^{\square}_{\Sigma_Q}$.*

*Proof.* We sketch a proof, which is based on one for the similar assertion in [75]. Firstly, suppose that $p > 2$. When $\zeta$ is distinct, for every minimal prime $\triangle$ of $\Lambda$, the Krull-dimension of irreducible $A^{\square}_{\Sigma_Q}/\triangle$ is

$$\begin{array}{rl} & 1 + r + (1 + 2[F:\mathbb{Q}] + \gamma_F) + 4|T| \\ = & 1 + (q - [F:\mathbb{Q}] - 1) + (1 + 2[F:\mathbb{Q}] + \gamma_F) + 4|T| \\ = & 1 + q + [F:\mathbb{Q}] + \gamma_F + 4|T|. \end{array}$$

On the other hand, the $A^{\square}_{\Sigma_Q}$-depth of $H^{\square}_{\Sigma_Q}/\triangle$ is *at least* the $\Lambda\hat{\otimes}R^{\square}[[\Delta_Q]]$-depth of $H^{\square}_{\Sigma_Q}/\triangle$; as $H^{\square}_{\Sigma_Q}/\triangle$ is free as a $(\Lambda\hat{\otimes}R^{\square}[[\Delta_Q]]/\triangle)$-module, the latter equals the Krull-dimension of $\Lambda\hat{\otimes}R^{\square}[[\Delta_Q]]$ which is greater than or equal to

$$\begin{aligned} & 1+(1+[F:\mathbb{Q}]+\gamma_F)+4|T|-1+q \\ = \;& 1+q+[F:\mathbb{Q}]+\gamma_F+4|T|. \end{aligned}$$

For a minimal prime $\triangle$ as above, it follows from Lemma 2.3 in [83] that $H^{\square}_{\Sigma_Q}/\triangle$ is a nearly faithful module over $A^{\square}_{\Sigma_Q}/\triangle$ when $\zeta$ is distinct. By Lemma 2.2, 1, [83], $H^{\square}_{\Sigma_Q}/(\triangle,\lambda)$ is a nearly faithful $A^{\square}_{\Sigma_Q}/(\triangle,\lambda)$-module when $\zeta$ is trivial. It then follows from Lemma 2.2, 2, [83], $H^{\square}_{\Sigma_Q}/\triangle$ is a nearly faithful $A^{\square}_{\Sigma_Q}/\triangle$-module when $\zeta$ is trivial. As this holds for any minimal prime $\triangle$, $H^{\square}_{\Sigma_Q}$ is a nearly faithful $A^{\square}_{\Sigma_Q}$-module when $\zeta$ is trivial.

On the other hand, $p$ and the generators of $J$ define a system of parameters of $A^{\square}_{\Sigma_Q}$, which indeed is a regular sequence since $A^{\square}_{\Sigma_Q}$ is Cohen-Macaulay. It therefore follows that $A^{\square}_{\Sigma_Q}/\lambda$ is reduced and the regularity of $\lambda$ then establishes that $A^{\square}_{\Sigma_Q}$ is reduced. It follows that $H^{\square}_{\Sigma_Q}$ is indeed a faithful module over $A^{\square}_{\Sigma_Q}$.

If $p=2$, the Krull-dimension of $A^{\square}_{\Sigma_Q}/\triangle$ is

$$\begin{aligned} & 1+(r-2-q)+(1+2[F:\mathbb{Q}]+\gamma_F)+4|T| \\ = \;& 1+(2q-[F:\mathbb{Q}]+1-2-q)+(1+2[F:\mathbb{Q}]+\gamma_F)+4|T| \\ = \;& 1+q+[F:\mathbb{Q}]+\gamma_F+4|T|; \end{aligned}$$

and the $A^{\square,\nabla}_{\Sigma_Q}$-depth of $H^{\square}_{\Sigma_Q}/\triangle$ is again *at least* the $\Lambda\hat{\otimes}R^{\square}[[\Delta_Q]]$-depth of $H^{\square}_{\Sigma_Q}/\triangle$. The rest follows similarly. □

**Corollary 17.** *$R^{\square}_{\Sigma_Q}/J\simeq R_\Sigma$ is reduced and $H_\Sigma$ is a faithful $R_\Sigma$-module. In particular, $R_\Sigma\simeq T_\Sigma$.*

*Proof.* See [75]. The outline of the proof in [75] is as follows. Firstly, since $H^{\square}_{\Sigma_Q}$ is a faithful $A^{\square}_{\Sigma_Q}$-module and $A^{\square}_{\Sigma_Q}/J$ is isomorphic to $R_\Sigma$, it follows from Lemma 2.2 in [83] that $H^{\square}_{\Sigma_Q}/J$ is a nearly faithful $R_\Sigma\simeq A^{\square}_{\Sigma_Q}/J$-module. Therefore it suffices to prove that $A^{\square}_{\Sigma_Q}/J$ is reduced. To prove that $A^{\square}_{\Sigma_Q}/J$ is reduced, one observes that $(A^{\square}_{\Sigma_Q}/J)[1/p]$ is generically reduced; indeed, one can make appeal to Lemma 18 below to prove that the localisation of $(A^{\square}_{\Sigma_Q}/J)[1/p]$ at its (any) minimal ideal, containing $J$ and $p$, is reduced. As $A^{\square}_{\Sigma_Q}/J$ is Cohen-Macaulay, so is $(A^{\square}_{\Sigma_Q}/J)[1/p]$, and therefore it is reduced. Since $A^{\square}_{\Sigma_Q}/J$ is noetherian local, one sees that $p$ is $A^{\square}_{\Sigma_Q}/J$-regular and therefore that $A^{\square}_{\Sigma_Q}/J$ is $p$-torsion free. As a result, $A^{\square}_{\Sigma_Q}/J$ injects into the reduced ring $(A^{\square}_{\Sigma_Q}/J)[1/p]$ and the reducedness of $A^{\square}_{\Sigma_Q}/J$ follows. The injectivity of the surjection $R_\Sigma\twoheadrightarrow T_\Sigma$ follows from the faithfulness of $H_\Sigma$ as an $R_\Sigma$-module. □

The following is due originally to Hu-Paškūnas [45]:

**Lemma 18.** *Let $R$ be a noetherian local ring and let $M$ be a faithful, Cohen-Macaulay, finitely generated $R$-module. Let $r,r_1,\ldots,r_N$ be a system of parameters of $R$, let $J$ denote the ideal generated by $r_1,\ldots,r_N$ and let $\overline{R}=R/J$ and $\overline{M}=M\otimes_R R/J$. Suppose that*

- *$M[1/r]$ is Cohen-Macaulay and faithful over $R[1/r]$,*

- $\overline{M}[1/r]$ *is a semi-simple* $\overline{R}[1/r]$*-module,*

- *for every prime ideal* $\Delta$ *in* $R[1/r]$ *which is the pre-image of a maximal ideal* $\mathfrak{m}$ *that lies in* $\mathrm{Supp}_{\overline{R}[1/r]}(\overline{M}[1/r])$*, the localisation of* $R[1/r]$ *at* $\Delta$ *is regular.*

*Then* $\overline{R}[1/r]$ *is reduced.*

*Proof.* See Lemma 19 in [75]. □

# 5 $\overline{\rho}$ is irreducible with soluble image

## 5.1 $\overline{\rho}$ is not induced from a character of an imaginary quadratic extension of $F$ in which every place of $F$ above $p$ does not split completely

Suppose $F$ satisfies the following conditions:

- $[F : \mathbb{Q}]$ is even,

- when

  $* \, p > 2$

  $*$ the restriction of $\overline{\rho}$ to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$ is reducible (while $\overline{\rho}$ remains irreducible), hence $\overline{\rho}$ is abelian when restricted to $\mathrm{Gal}(\overline{F}/F^+)$ for the quadratic extension $F^+$ over $F$ in $F(\zeta_p)$,

  $*$ $F^+$ is imaginary over $F$,

  hold simultaneously, suppose that not every place in $S_p$ splits completely in $F^+$,

- when

  $* \, p = 2$

  $*$ $\overline{\rho}$ has soluble image (while $\overline{\rho}$ remains irreducible), hence $\overline{\rho}$ is abelian when restricted to $\mathrm{Gal}(\overline{F}/E)$ for a quadratic extension $E$ over $F$,

  $*$ $E$ is imaginary over $F$

  hold simultaneously, suppose that not every place in $S_p$ splits completely in $E$,

**Definition**. A prime ideal $\Gamma$ of $R_\Sigma$ is said to be *pro-modular* if $\Gamma$ contains the kernel of the surjective homomorphism $r = r(\overline{\rho}) : R_\Sigma \to T_\Sigma$; in which case, $R_\Sigma \to R_\Sigma/\Gamma$ factors as

$$\begin{array}{ccc} R_\Sigma & \twoheadrightarrow & R_\Sigma/\Gamma \\ \downarrow & & \uparrow \\ T_\Sigma & \overset{r^{-1}}{\to} & R_\Sigma/\ker r \end{array}$$

where the varical maps are both surjective and it is pro-modular in the sense of [81].

**Definition**. A prime $\Gamma$ of $R_\Sigma$ is said to be admissible if is is of dimension 1 and contains $p$ and if we let $\rho = \rho_\Gamma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$ where $R = R_\Gamma$ denotes the normalisation of the quotient of $R$ by $\Gamma$ in its field $K$ of fractions,

- $\Gamma$ is of dimension 1 and contains $p$; enlarging $\mathcal{O}$ if necessary, we may assume that $R$ is isomorphic to a power series ring over $\mathbb{F}$ with a single variable, say $\pi$),
- $\rho \otimes_R \overline{K}$ is irreducible,
- $\det \rho$ is of finite order,
- if $p > 2$ (resp. $p = 2$), then $\rho$ is not $F^+$-dihedral (resp. $\rho$ is not dihedral),
- for every v in $S_p$ (resp. $S_R$) , the restriction $\rho_v$ of $\rho$ at v is reducible with distinct diagonal characters (resp. is trivial),
- $\Gamma$ is pro-modular.

We firstly assume that an admissible prime $\Gamma$ of $R_\Sigma$ exists– this will be proved in Proposition 32.

Following the discussion at the beginning of Section 7 in [80] (and Section 4.6 in [89]), we may, and will, replace $\Lambda$ by its finite faithfully flat extension in such a way that the natural map $\Lambda \to R_\Sigma$ gives rise to an isomorphism modulo $\Gamma$ and is isomorphic to the formal power series ring $\mathbb{F}[[\pi]]$, and the induced map on completed-localisation at $\Gamma$ also defines an isomorphism on their respective residue field (isomorphic to $\mathbb{F}[[\pi]]$).

## 5.2 Selmer groups

Let $R = \mathbb{F}[[\pi]]$ and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$.

**Definition**. Given a module over $R$, by the *order* of an element of the module, we shall mean the smallest power of $\pi$ that annihilates it.

Let $F_N = F(\zeta_{p^N})$ and $F_\infty$ denote the union of the $F_N$'s. Following [22] and [89], we define (dual) Selmer groups for $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$ that we are interested in: Fix a deformation data

$$\Sigma = (S, T, (L_v)_{v \in S})$$

where it is assumed that $S$ contains $S_p$ and $S_\infty$, and while $T = S - S_\infty$ as before.

Suppose that $S_{Q,N}$ is disjoint from $S$ as in previous sections. Following [89] Section 5.2, define $H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes_R R/\pi^r)$ to be the cohomology group defined as in Section 2 in [75] with $R/\pi^r$ in place of $\mathbb{F}$, $S \cup S_{Q,N}$ in place of $S$ (we write subspaces $L_v^{(r)} \subset H^1(D_v, \mathrm{ad}\,\rho \otimes_R R/\pi^r)$ defined analogously over $R/\pi^r$ at $S \cup S_{Q,N}$). Similarly define $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes_R R/\pi^r)$ following Section 2 in [75].

As $r \geq 1$ varies, the $H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes_R R/\pi^r)$ (resp. $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes_R R/\pi^r)$) defines a direct system system, and let

$$H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes_R K/R) = \lim_{r \to} H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes_R R/\pi^r)$$

$$(\text{resp. } H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes_R K/R) = \lim_{r \to} H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes_R R/\pi^r))$$

denote the limit.

**Proposition 19.** *Fix $r$.*

$$H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes_R R/\pi^r) \simeq H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes_R \otimes_R K/R)[\pi^r]$$

$$H^1_{\Sigma^{\perp}_{Q,N}}(F, \operatorname{ad}\rho(1) \otimes_R R/\pi^r) \simeq H^1_{\Sigma^{\perp}_{Q,N}}(F, \operatorname{ad}\rho(1) \otimes_R \otimes_R K/R)[\pi^r]$$

*Proof.* By definition, for every integer $r \geq s$, we have a commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \in T} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^s) & & \bigoplus_{v \in T} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^r) & & \bigoplus_{v \in T} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^{r-s}) \\
\oplus & \longrightarrow & \oplus & \xrightarrow{\pi^s} & \oplus \\
\bigoplus_{v \in S_\infty} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^s)/N_v^{(s)} & & \bigoplus_{v \in S_\infty} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^r)/N_v^{(r)} & & \bigoplus_{v \in S_\infty} H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^{r-s})/N_v^{(r-s)} \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^s) & \longrightarrow & H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^r) & \xrightarrow{\pi^s} & H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^{r-s}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^s) & \longrightarrow & H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^r) & \xrightarrow{\pi^s} & H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^{r-s})
\end{array}$$

where the direct sums in the first row range over $T$.

We firstly show that $\ker\left(H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^s) \to H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^r)\right) = 0$. This follows from observing

$$0 \to H^0(F_S, \operatorname{ad}\rho \otimes R/\pi^s) \to H^0(F_S, \operatorname{ad}\rho \otimes R/\pi^r) \to H^0(F_S, \operatorname{ad}\rho \otimes R/\pi^{r-s}) \to H^1(F_S, \operatorname{ad}\rho \otimes R/\pi^s) \to \cdots$$

that the boundary map is indeed zero, for it is isomorphic to $0 \to R/\pi^s \to R/\pi^r \xrightarrow{\pi^s} R/\pi^{r-s} \to 0$. Granted, since $\ker\left(H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^s) \to H^0(D_v, \operatorname{ad}\rho \otimes R/\pi^r)\right) = 0$ for v in $T$, and similarly at v in $S_\infty$, it follows that

$$\ker\left(H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^s) \to H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^r)\right) = 0,$$

thereby

$$H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^s) \simeq H^1_{\Sigma_{Q,N}}(F, \operatorname{ad}\rho \otimes R/\pi^r)[\pi^s].$$

For the second assertion, we similarly observe for $r \geq s$, we have a commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\Sigma^{\perp}_{Q,N}}(F, \operatorname{ad}\rho(1) \otimes R/\pi^s) & \to & H^1_{\Sigma^{\perp}_{Q,N}}(F, \operatorname{ad}\rho(1) \otimes R/\pi^r) & \to & H^1_{\Sigma^{\perp}_{Q,N}}(F, \operatorname{ad}\rho(1) \otimes R/\pi^{r-s}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(F_S, \operatorname{ad}\rho(1) \otimes R/\pi^s) & \to & H^1(F_S, \operatorname{ad}\rho(1) \otimes R/\pi^r) & \to & H^1(F_S, \operatorname{ad}\rho(1) \otimes R/\pi^{r-s}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \in S_{Q,N}} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^s)/L_v^{(s)\perp} & \to & \bigoplus_{v \in S_{Q,N}} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^r)/L_v^{(r)\perp} & \to & \bigoplus_{v \in S_{Q,N}} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^{r-s})/L_v^{(r-s)\perp} \\
\oplus & & \oplus & & \oplus \\
\bigoplus_{v \in S_\infty} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^s)/L_v^{(s)\perp} & & \bigoplus_{v \in S_\infty} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^r)/L_v^{(r)\perp} & & \bigoplus_{v \in S_\infty} H^1(D_v, \operatorname{ad}\rho(1) \otimes R/\pi^{r-s})/L_v^{(r-s)\perp}
\end{array}$$

Observe that for v in $S_{Q,N} \cup S_\infty$,

$$\ker\left(H^0(D_v, \mathrm{ad}\,\rho(1) \otimes R/\pi^s)/M_v^{(s)\perp} \to H^0(D_v, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)/M_v^{(r)\perp}\right) = 0,$$

and

$$\ker\left(H^1(D_v, \mathrm{ad}\,\rho(1) \otimes R/\pi^s)/L_v^{(s)\perp} \to H^1(D_v, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)/L_v^{(r)\perp}\right) = 0.$$

It then follows that $\ker\left(H^1_{\Sigma^\perp_{Q,N}}(F_S, \mathrm{ad}\,\rho(1) \otimes R/\pi^r) \to H^1_{\Sigma^\perp_{Q,N}}(F_S, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)\right)$ is isomorphic to

$$\ker\left(H^1(F_S, \mathrm{ad}\,\rho(1) \otimes R/\pi^r) \to H^1(F_S, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)\right).$$

On the other hand, the latter is zero as in the proof of the first assertion. It therefore follows that

$$H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)[\pi^s] \simeq H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes /\pi^r)[\pi^s].$$

Since

$$\lim_{\to r} H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes R/\pi^r)[\pi^s] \simeq H^1_{\Sigma_{Q,N}}(F, \mathrm{ad}\,\rho \otimes K/R)[\pi^s]$$

and

$$\lim_{\to r} H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes R/\pi^r)[\pi^s] \simeq H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes K/R)[\pi^s],$$

the assertions follow. □

Everything we need is proved in [1]. We give it a slightly different narrative to be consistent with our approach. The underling space of $\mathrm{ad}\,\rho$ is the set of 2-by-2 matrices over $R$ and let $Z$ be the normal subgroup of scalar matrices.

We have

$$\begin{array}{ccccccc} & & & & & & H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho(1)) \\ & & & & & & \downarrow \\ 0 & \to & H^1(\mathrm{Gal}(F_N/F), Z(1)) & \to & H^1(F_S, \mathrm{ad}\,\rho(1)) & \to & H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\rho(1)) \\ & & & & & & \downarrow \\ & & & & & & H^1(\mathrm{Gal}(F_S/K_N), \mathrm{ad}\,\rho(1)) \end{array}$$

where the kernel $H^1(\mathrm{Gal}(F_N/F), Z(1))$ is zero if $p > 2$ and is of rank 2 over $R$ if $N > N_{\mathrm{kw}}$ (Lemma 3.2.3 in [1]; the result is the $R/\pi^r$-analogue of the observations earlier $\dim_{\mathbb{F}} \ker(H^1(F_S, \mathrm{ad}\,\overline{\rho}) \to H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\overline{\rho})) = 2$ in the proof of Lemma 13); and $\mathrm{Gal}(K_N/F_N)$ is isomorphic to the image of $\rho$.

We will be interested in a non-trivial class in $H^1(F_S, \mathrm{ad}\,\rho)$ whose image in $H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\rho(1))$ is non-trivial (if $p > 2$, then any class in $H^1(F_S, \mathrm{ad}\,\rho)$ will be).

Let $G = \mathrm{GL}_2$ and $\overline{G} = \mathrm{SL}_2$. Let $\Delta \subset G(R)$ denote the image of $\rho$ and $\overline{\Delta}$ denote $\Delta \cap \overline{G}(R)$. If $\overline{\rho}$ is not dihedral (resp. is dihedral), then it follows from $\rho$ being '$p$-distinguished' with its determinant of finite order (resp. from $\rho$ being not dihedral) that $\Delta \hookrightarrow G(R) \to G(\mathbb{F})$ has non-trivial kernel, and it follows from Proposition 1.7.5 in [1] that $\overline{\Delta}$ is Zariski dense in $\overline{G}$ over $K$. Since the image by $\rho$ of a complex conjugation defines a non-trivial unipotent element in $\Delta$, it follows from Proposition 3.1.2 in [1], which is based on Theorem 0.2 in [71], that there exists a sub-extension $L$ of $K$ of finite index and an algebraic group $\overline{H}$ over $L$ such that $\overline{\Delta}$ is thought of as an open compact subgroup of $\overline{H}(L)$ up to conjugation in $G(L)$. By abuse of notation, we write $\overline{G}$ (resp. $K$) for $\overline{H}$ (resp. $L$). With these in mind, we think of $\overline{\Delta}$ as an open compact subgroup of $\overline{G}(R)$.

**Lemma 20.** *There exists a non-negative integer $e$ such that every non-trivial $\mathrm{Gal}(F_S/F_N)$-stable submodule of $(\mathrm{ad}\,\rho/Z) \otimes R/\pi^r$ annihilated by $\pi^s$, where $s \leq e$, contains an element whose trace in $R/\pi^r$ does not lie in $\pi^e(R/\pi^r)$.*

*Proof.* This is proved in Lemma 3.2.5 in [1].

**Remark**. This lemma plays the role of Lemma 6.5 in [80] when $p > 2$ and $\rho$ is not dihedral, and Lemma 2.5.2 in [79] when $p > 2$ and $\rho$ is dihedral but not $F^+$-dihedral.

On the other hand, it follows from $H^0(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z) = 0$ (Lemma 3.1.4 in [1]) that we have an exact sequence

$$0 \to H^1(\mathrm{Gal}(K_N/F_N), Z) \to H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho) \to H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z) \to \cdots,$$

where Lemma 3.5.1 in [1] proves that the image of $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho) \to H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z)$ is at most rank 1 over $R$, and so is the image of $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho \otimes R/\pi^r) \to H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$ over $R/\pi^r$. Indeed, it is possible to prove

**Lemma 21.** *There exists a non-negative integer $s$ such that $\pi^s$ annihilates the image of $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho \otimes R/\pi^r)$ in $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$.*

*Proof.* Let $\overline{\Delta}_e$ denote the principal congruence subgroup of matrices in $\overline{\Delta} \subset \overline{G}(R)$ which are congruent modulo $\pi^e$ to 1 in $\overline{G}(R/\pi^e)$. Since $\overline{\Delta}$ is open compact in $\overline{G}(R)$, there exists a sufficiently large $e$ such that $\Delta = \mathrm{Gal}(K_N/F_N)$ contains $\overline{\Delta}_e$.

Firstly we observe that the assertion is equivalent to establishing that $\pi^s$ annihilates the cokernel of $H^1(\Delta, Z \otimes R/\pi^r) \to H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r)$. This, in turn, is equivalent to establishing that there exists a non-negative integer $s$ such that $\pi^s$ annihilates the cokernel of the composite

$$H^1(\Delta, Z \otimes R/\pi^r) \to H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r) \to H^1(\overline{\Delta}_e, \mathrm{ad}\,\rho/Z \otimes R/\pi^r).$$

Since $H^1(\Delta, Z \otimes R/\pi^r) \to H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r)$ is injective, it suffices to show that the kernel of the composite $H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r) \to H^1(\overline{\Delta}_e, \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$ is $H^1(\Delta, Z \otimes R/\pi^r)$. If $\overline{N}$ is an open *normal* subgroup of $\overline{\Delta}$ contained in $\overline{\Delta}_e$, then the subspace of $\overline{N}$-invariants of $\mathrm{ad}\,\rho/Z$ is trivial, and therfore the composite

$$H^1(\Delta, \mathrm{ad}\,\rho/Z \otimes R/\pi^r) \to H^1(\overline{\Delta}_e, \mathrm{ad}\,\rho/Z \otimes R/\pi^r) \to H^1(\overline{N}, \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$$

is injective (since the $\overline{N}$-invariants of $\mathrm{ad}\,\rho/Z$ is trivial); it therefore follows that $H^1(\Delta, \mathrm{ad}\,\rho/Z \otimes R/\pi^r) \to H^1(\overline{\Delta}_e, \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$ is injective. The kernel of $H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r) \to H^1(\Delta, \mathrm{ad}\,\rho/Z \otimes R/\pi^r) \to H^1(\overline{\Delta}_e, \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$ is computed by the kernel of $H^1(\Delta, \mathrm{ad}\,\rho \otimes R/\pi^r) \to H^1(\Delta, \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$ which is $H^1(\Delta, Z \otimes R/\pi^r)$.

Let $\phi$ be a class in the image of $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho \otimes R/\pi^r)$ in $H^1(\mathrm{Gal}(K_N/F_N), \mathrm{ad}\,\rho/Z \otimes R/\pi^r)$. Step 1, Step 2 and Step 3 in the proof of Lemma 3.1.6 in [1] then show that there exists a non-negative integer $s$ such that $\pi^s\phi$ is uniquely determined by its restriction to $\overline{\Delta}_e$, and that it is indeed zero. ☐

**Remark**. When $p > 2$ and $\rho$ is not dihedral, this is proved in Lemma 6.9 in [80]. If $p > 2$ and $\rho$ is dihedral but not $F^+$-dihedral, this is proved in Lemma 2.5.3 in [79].

**Proposition 22.** *There exists an integer $e > 0$ such that, given a class $\phi$ in $H^1(F_S, \operatorname{ad}\rho(1) \otimes R/\pi^r)$ of order $s(\phi) > e$ which maps to a non-trivial class in $H^1(\operatorname{Gal}(F_S/F_N), \operatorname{ad}\rho(1) \otimes R/\pi^r)$ (where $N > N_{\mathrm{kw}}$ if $p = 2$) of order $s(\phi)$, then there exists a place* v *of $F$ such that*

- $\mathbf{N}_{F/\mathbb{Q}}\mathrm{v} \equiv 1$ *mod* $p^N$,
- $\overline{\rho}$ *is unramified at* v *and* $\overline{\rho}(\mathrm{Frob}_{\mathrm{v}})$ *has distinct eigenvalues,*
- *the trace of the* $\mathrm{Frob}_{\mathrm{v}}$*-equivariant projection to a chosen eigenspace of* $\phi(\mathrm{Frob}_{\mathrm{v}}) \subset \operatorname{ad}\rho \otimes R/\pi^r$ *has order* $s(\phi) - e$.

*Proof.* Let $\phi_N$ denote the image of $\phi$ in $H^1(\operatorname{Gal}(F_S/F_N), \operatorname{ad}\rho(1) \otimes R/\pi^r)$, and $\varphi_N$ denote the image of $\phi_N$ in the $\operatorname{Gal}(K_N/F_N)$-invariant subspace of $H^1(\operatorname{Gal}(F_S/K_N), \operatorname{ad}\rho(1) \otimes R/\pi^r)$.

It suffices to prove that there is a non-negative integer $e$ such that given a non-trivial class $\phi_N$ in $H^1(\operatorname{Gal}(F_S/F_N), \operatorname{ad}(1) \otimes R/\pi^r)$, then there is $\gamma$ in $\operatorname{Gal}(F_S/F_N)$ such that $\overline{\rho}$ is unramfied at $\gamma$. If an eigenvalue of $\overline{\rho}(\gamma)$ is chosen, then the trace $\tau(\phi_N)$ of the $\gamma$-equivariant projection onto the corresponding eigenspace has order at least $s(\phi) - e$.

Suppose that $\varphi_N$ is trivial; in which case, it defines a class in $H^1(\operatorname{Gal}(K_N/F_N), \operatorname{ad}\rho \otimes R/\pi^r)$. If it maps trivially to $H^1(\operatorname{Gal}(K_N/F_N), \operatorname{ad}\rho/Z \otimes R/\pi^r)$, then it defines an element of $H^1(\operatorname{Gal}(K_N/F_N), Z \otimes R/\pi^r)$ and the assertion is proved in (a) of Lemma 3.2.9 of [1]. As $\phi_N$ has exact order $s(\phi)$, one can choose an element $\gamma$ of $\operatorname{Gal}(F_S/F_N)$ such that $\phi_N(\gamma)$ has order $s(\phi)$ in $R/\pi^r$. Thinking of $\phi_N$ as a homomorphism from $\operatorname{Gal}(F_S/F_N)$ to $Z(1) \otimes R/\pi^r$, one may and will find an element $g$ in $\ker \phi_N$ such that $\overline{\rho}(g\gamma)$ has distinct eigenvalues. Choose one eigenvalue. If $\tau$ denotes the map taking the trace of the $\gamma$-equivariant projection onto the eigenspace for the chosen eigenvalue, $\tau(\phi_N(g\gamma)) = \tau(\phi_N(g)) + \tau(\phi_N(\gamma)) = \tau(\phi_N(\gamma))$ has order $s(\phi)$ by construction. The place v corresponding to $g\gamma$ proves the assertion.

Suppose that $\varphi_N$ is trivial but defines a non-trivial class in $H^1(\operatorname{Gal}(K_N/F_N), \operatorname{ad}\rho/Z \otimes R/\pi^r)$.It follows from Lemma 21 that there exists a positive integer $s$ such that $\pi^s\phi_N$ defines a class in $H^1(\operatorname{Gal}(K_N/F_N), Z \otimes R/\pi^r)$ of order at least $s(\phi) - s$. An argument similar to the one seen above proves the assertion.

Suppose that $\varphi_N$ is non-trivial; this is proved by (c) of the proof of Lemma 3.2.9 in [1]. It defines either a trivial, or a non-trivial, class in $H^1(\operatorname{Gal}(F_S/K_N), \operatorname{ad}\rho/Z \otimes R/\pi^r)$. If it defines a trivial class, then $\varphi_N$ defines a class in $H^1(\operatorname{Gal}(F_S/K_N), Z \otimes R/\pi^r)$ and an argument similar to the one seen above proves the assertion. We therefore assume that we have a non-trivial class in $H^1(\operatorname{Gal}(F_S/K_N), \operatorname{ad}\rho/Z \otimes R/\pi^r)$. Since $\pi^s$ annihilates $H^1(\operatorname{Gal}(K_N/F_N), \operatorname{ad}\rho/Z)$ by Lemma 21, it follows that $\varphi_N$ has order at least $s(\phi) - s$. Thinking of $\varphi_N$ as a homomorphism $\varphi_N$ : $\operatorname{Gal}(F_S/K_N) \to \operatorname{ad}\rho/Z \otimes R/\pi^r$ whose image is $\operatorname{Gal}(F_S/F_N)$-invariant, it follows from Lemma 20 that the image contains an element $g$ whose trace is of order at least $s(\phi) - e$.

We then find $\gamma$ in $\operatorname{Gal}(F_S/F_N)$ such that $\overline{\rho}(\gamma)$ has distinct eigenvalues. Choose one of the eigenvalues and let $\tau$ denote the corresponding map as before. If $\tau(\phi(\gamma))$ has order at least $s(\phi) - e$, then we are done. If not, $\tau(\phi(g\gamma)) = \tau(\phi(\gamma)) + \tau(\phi(g))$ has order at least $s(\phi) - e$. By the Chebotarev density theorem, we find v such that $\mathrm{Frob}_{\mathrm{v}}$ defines $\gamma$ or $\gamma g$. ☐

**Corollary 23.** *For every $N$ when $p > 2$ and for every $N > N_{\text{kw}}$ if $p = 2$, there exists a set $S_{Q,N}$ of primes v as above such that*

- $|S_{Q,N}| = q$ *where* $q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1))$ *if* $p > 2$ *and* $q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 2$ *if* $p = 2$;
- *when* $p > 2$, $H^1_{\Sigma^\perp_{Q,N}}(F_S, \mathrm{ad}\rho(1) \otimes_R L/R)$ *is a finite* $R$*-module bounded independently of* $N$*, while when* $p = 2$,
$$H^1_{\Sigma^\perp_{Q,N}}(F_S, \mathrm{ad}\,\rho(1) \otimes_R R/\pi^r) \asymp (R/\pi^r)$$
*holds for every* $r$.
- $$H^1_{\Sigma_{Q,N}}(F_S, \mathrm{ad}\,\rho \otimes_R K/R) \simeq (K/R)^r \oplus X_{\Sigma_{Q,N}}$$
*holds, where* $X_{\Sigma_{Q,N}}$ *is a finite* $R$*-module with* $|X_{\Sigma_{Q,N}}|$ *bounded independently of* $N$*, and where* $r = q - [F : \mathbb{Q}] - 1$ *(resp.* $r = 2q - [F : \mathbb{Q}] + 1$*) if* $p > 2$ *(resp.* $p = 2$*).*

*Proof.* Repeatedly apply Lemma 12 (resp. Lemma 13) to a set of $q$ classes, of order $r$, in $H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\rho(1)\otimes R/\pi^r)$ which map to non-trivial classes in $H^1(\mathrm{Gal}(F_S/F_N), \mathrm{ad}\,\rho \otimes R/\pi^r))$ if $p > 2$ (resp. $p = 2$). In the light of Proposition 19, this proves that $H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho \otimes_R K/R)$ is a finite $R$-module of order independent of $N$ when $p > 2$; when $p = 2$, the second assertion further requires arguments as in Lemma 13.

To deduce the third assertion, one observes that an analogue of Proposition 5 holds with $\rho \otimes R/\pi^r$ in place of $\overline{\rho}$ and one therefore sees that $\mathrm{rk}\, H^1_{\Sigma_{Q,N}}(F_S, \mathrm{ad}\,\rho \otimes_R R/\pi^r)$ (where by rk, we mean the exponent, with respect to $r$, of the cardinality of what follows) is computed by

$$\mathrm{rk}\, H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\rho(1) \otimes_R R/\pi^r) - 1 - [F : \mathbb{Q}] + \sum_{\mathrm{v} \in S_{Q,N}} 1 = q - [F : \mathbb{Q}] - 1$$

where $|S_{Q,N}| = q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1))$ if $p > 2$, and

$$\begin{aligned} & \mathrm{rk}\, H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\rho(1) \otimes_R R/\pi^r) - 1 - [F : \mathbb{Q}] + \textstyle\sum_{\mathrm{v} \in S_{Q,N}} 2 \\ = \;& 2 - 1 - [F : \mathbb{Q}] + 2q = 2q - [F : \mathbb{Q}] + 1 \end{aligned}$$

where $|S_{Q,N}| = q = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - \dim_{\mathbb{F}} H^1_{\Sigma^\perp_{Q,N}}(F, \mathrm{ad}\,\overline{\rho}(1)) = \dim_{\mathbb{F}} H^1_{\Sigma^\perp}(F, \mathrm{ad}\,\overline{\rho}(1)) - 2$ if $p = 2$. □

## 5.3 Patching and localised $R = T$

**Definition**. For a ring $R$ and a prime ideal $\Gamma$, let $R^\Gamma$ denote the completion at the maximal ideal of the localisation of $R$ at $\Gamma$.

The universal representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R_\Sigma)$ specialises to a representation

$$\rho_\Gamma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$$

over $R = \mathbb{F}[[\pi]]$. Let $\boldsymbol{\mu}_{\Sigma_{Q,N}}$ denote the kernel of $R^\square_{\Sigma_{Q,N}} \to R_\Sigma \to R$. Let $\boldsymbol{\mu}$ denote the pre-image of $\boldsymbol{\mu}_{\Sigma_{Q,N}}$ in $A^\square_\Sigma$ by the map $A^\square_\Sigma = A^\square_{\Sigma_{Q,N}} \to R^\square_{\Sigma_{Q,N}}$.

By slight abuse of notation, let $A_{\Sigma}^{\square,\Gamma}$ (resp. $R_{\Sigma_{Q,N}}^{\square,\Gamma}$) denote the completion of the localisation of $A_{\Sigma}^{\square}$ (resp. $R_{\Sigma_{Q,N}}^{\square}$) at $\boldsymbol{\mu}$ (resp. $\boldsymbol{\mu}_{\Sigma_{Q,N}}$). We let $A_{\Sigma_Q}^{\square,\Gamma} = A_{\Sigma}^{\square,\Gamma}[[X_1, \ldots, X_r]]$ and, if $p = 2$, we furthermore let $A_{\Sigma_Q}^{\square,\nabla,\Gamma} = A_{\Sigma}^{\square,\Gamma}[[X_1, \ldots, X_{r-(q+2)}]]$.

**Proposition 24.** *Let $r$ be an integer defined in Corollary 23. If $p > 2$ (resp. $p = 2$), then, for every $N$ (resp. for every $N > N_{\mathrm{kw}}$), there is an isomorphism of $R$-mdoules between $\boldsymbol{\mu}_{\Sigma_{Q,N}}/(\boldsymbol{\mu} + \boldsymbol{\mu}_{\Sigma_{Q,N}}^2)$ and $R^r \oplus X_{\Sigma_{Q,N}}$ for some $R$-module $X_{\Sigma_{Q,N}}$ with $|X_{\Sigma_{Q,N}}|$ bounded independently of $N$.*

*Proof.* One can argue as in the proof of Corollary 5.7 in [89] that

$$\mathrm{Hom}_R(\boldsymbol{\mu}_{\Sigma_{Q,N}}/(\boldsymbol{\mu} + \boldsymbol{\mu}_{\Sigma_{Q,N}}^2), K/R) \simeq H_{\Sigma_{Q,N}}^1(F, \mathrm{ad}\,\boldsymbol{\rho} \otimes_R K/R)$$

as $R$-modules. Hence the assertion follows from Corollary 23. □

We may and will let the surjective $A_{\Sigma}^{\square}$-algebra homomorphism

$$A_{\Sigma_Q}^{\square,\Gamma} \longrightarrow R_{\Sigma_{Q,N}}^{\square,\Gamma}$$

be defined such that the $r$ formal variables in $A_{\Sigma_Q}^{\square}$ map to $\boldsymbol{\mu}_{\Sigma_{Q,N}}$ and define the maximal $R$-free quotient of $\boldsymbol{\mu}_{\Sigma_{Q,N}}/(\boldsymbol{\mu} + \boldsymbol{\mu}_{\Sigma_{Q,N}}^2)$. When $p = 2$, this furthermore induces

$$A_{\Sigma_Q}^{\square,\nabla,\Gamma} \longrightarrow R_{\Sigma_{Q,N}}^{\square,\Gamma}/\nabla_{Q,N}.$$

By patching, we have $A_{\Sigma_Q}^{\square}\hat{\otimes}\Lambda\hat{\otimes}_{\mathscr{O}}R^{\square}[[\Delta_Q]]$-module $H_{\Sigma_Q}^{\square}$ which is free over $\Lambda_Q^{\square}$. Let $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}$ denote the completed localisation of $\Lambda\hat{\otimes}_{\mathscr{O}}R^{\square}[[\Delta_Q]]$ at the pre-image of $\Gamma$ in $\Lambda\hat{\otimes}_{\mathscr{O}}R^{\square}[[\Delta_Q]]$; similarly define the completed localisations $R_{\Sigma_Q}^{\square,\Gamma}$, $H_{\Sigma_Q}^{\square,\Gamma}$, and $A_{\Sigma_Q}^{\square,\Gamma}$ at the respective images of $\Gamma$.

$$\begin{array}{ccccccc}
 & & \left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma} & & & & \\
 & & \downarrow & & & & \\
A_{\Sigma_Q}^{\square,\Gamma} & \longrightarrow & R_{\Sigma_Q}^{\square,\Gamma} & \longrightarrow & T_{\Sigma_Q}^{\square,\Gamma} & \subset & \mathrm{End}(H_{\Sigma_Q}^{\square,\Gamma}) \\
 & & \downarrow & & \downarrow & & \\
 & & R_{\Sigma}^{\Gamma} & \longrightarrow & T_{\Sigma}^{\Gamma} & &
\end{array}$$

if $p > 2$. We obtain a similar diagram with $R_{\Sigma_Q}^{\square,\Gamma}/\nabla_Q$ (resp. $A_{\Sigma_Q}^{\square,\nabla,\Gamma}$) in place of $R_{\Sigma_Q}^{\square,\Gamma}$ (resp. $A_{\Sigma_Q}^{\square,\Gamma}$) if $p = 2$.

**Proposition 25.** *Suppose the conditions in the preceding lemma. Suppose furthermore that, for every $N$ if $p > 2$, or every $N > N_{\mathrm{kw}}$ when $p = 2$, there exists a set $S_{Q,N}$ as in Corollary 23. Then $H_{\Sigma}^{\Gamma}$ is a faithful $R_{\Sigma}^{\Gamma}$-module. As a result, $R_{\Sigma}^{\Gamma} \twoheadrightarrow T_{\Sigma}^{\Gamma}$ is an isomorphism.*

We need a few lemmas.

**Lemma 26.**

- *$H_{\Sigma_Q}^{\square,\Gamma}$ is a free module over $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}$.*
- *$H_{\Sigma_Q}^{\square,\Gamma}/J \simeq H_{\Sigma}^{\Gamma}$.*
- *$A_{\Sigma_Q}^{\square,\Gamma} \to R_{\Sigma_Q}^{\square,\Gamma}$ (resp. $A_{\Sigma_Q}^{\square,\nabla,\Gamma} \to R_{\Sigma_Q}^{\square,\Gamma}/\nabla_Q$) is surjective.*

*Proof of the lemma.* The first assertion is standard. The second assertion can be proved as in Lemma 15.

To prove the third, it suffices to establish that the relative tangent space vanish after $\otimes_R K$. By Proposition 24, the patching argument 'spends' the free $R$-part of $\boldsymbol{\mu}_{\Sigma_{Q,N}}/(\boldsymbol{\mu}+\boldsymbol{\mu}^2_{\Sigma_{Q,N}})$ in taking the limit, and the relative tangent space is consequently a finite $R$-torsion module. This evidently turns zero when $\otimes_R K$. □

Suppose that $\rho = \rho_\Gamma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R)$ is

- reducible at every place in $S_p$ with distinct diagonal characters,
- trivial at every place in $S_R$,
- unramified at every place v in $S_L$ and $\rho(\mathrm{Frob}_v)$ is a scalar in $1+\mathfrak{m}_R$,
- unramified at every place v in $S_A$ and $\rho(\mathrm{Frob}_v)$ has equal (resp. distinct) eigenvalues if $p>2$ (resp. $p=2$).

One can make a finite totally real soluble base change to ascertain $\rho$ is 'Steinberg' at every place in $S_L$ without further expenditure of effort (the image by $\rho$ of a generator of the $p$-part of the tame inertia subgroup at v in $S_L$ is unipotent, hence of finite $p$-power order; while the image of the inertia subgroup at v is finite), but it is not possible to do so similarly at $S_R$. It is for this reason we 'prescribe' $\Gamma$ with the property that $\rho_\Gamma$ is trivial at every place in $S_R$– it is under these assumptions that one can establish $A^\Gamma_\Sigma$ and $R^\Gamma_\Sigma$, rather than their quotients $A_\Sigma/\Gamma$ and $R_\Sigma/\Gamma$, satisfy ring-theoretic properties one needs to prove a localised $R=T$ theorem.

**Lemma 27.** *If the characters $\zeta$ at $S_R$ are distinct (resp. trivial), then for every minimal ideal $\triangle$ of $\Lambda$, the quotient $A^{\square,\Gamma}_\Sigma/\triangle$ is $\mathscr{O}$-flat and geometrically irreducible (resp. equidimensional) of dimension*

$$q+1+[F:\mathbb{Q}]+\gamma_F+4|T|,$$

*where $T=S_p\cup S_R\cup S_L\cup S_A$. Furthermore, when $\zeta$ is trivial and $L$ is sufficiently large,*

- *$A^{\square,\Gamma}_\Sigma[1/p]$ is regular, $A^{\square,\Gamma}_\Sigma$ is Cohen-Macaulay, and $A^{\square,\Gamma}_\Sigma/\lambda$ is generically reduced.*
- *every minimal prime of $A^{\square,\Gamma}_\Sigma/(\triangle,\lambda)$ contains a unique minimal prime of $A^{\square,\Gamma}_\Sigma/\triangle$.*

**Remark**. The same set of assertions hold with $A^\square_{\Sigma_Q}$ in place of $A^\square_\Sigma$.

*Proof.* This follows the proof of Lemma 3.4 in [89], but we shall sketch a proof. Firstly, checking properties of $A^\square_\Sigma$ remain unchanged in passing to the faithful base change, as alluded at the end of Section 5.1 will be left as an exercise for readers.

For every v in $T$, we let $A^\square_v$ denote the quotient of $R_v$ as defined in Section with maximal ideal $\mathfrak{m}^\square_v$. Let

$$A^\square_{\Sigma_p\cup\Sigma_A} = A^\square_{\Sigma_p}\hat{\otimes}\hat{\bigotimes}_{v\in S_A}A^\square_v,$$

where, to recall, $A^\square_{\Sigma_p}$ denotes $(\hat{\bigotimes}_{v\in S_p}A^\square_v)\hat{\otimes}_{\mathscr{O}[[\Delta_p\times\Delta_p]]}\Lambda$ and

$$A^\square_{\Sigma_L\cup\Sigma_R} = \hat{\bigotimes}_{v\in S_L\cup S_R}A^\square_v.$$

Twisting by the inverse of an unramified character of $D_v$ which sends $\mathrm{Frob}_v$ to the scalar value of $\rho_\Gamma(\mathrm{Frob}_v)$ at every v in $S_L$ gives rise to a map

$$\zeta : A_\Sigma^\square \to A_\Sigma^\square.$$

Let $\Gamma^\zeta$ denote the image of $\Gamma = \Gamma^\Sigma$ by $\zeta$. We then have

$$A_\Sigma^{\square,\Gamma} \simeq A_\Sigma^{\square,\Gamma^\zeta}.$$

We let $\Gamma^\zeta_{\Sigma_p \cup \Sigma_A}$ denote $\ker(A_\Sigma^\square \to A_{\Sigma_p\cup\Sigma_A}^\square \to R)$ and $\Gamma^\zeta_{\Sigma_L\cup\Sigma_R} = \ker(A_\Sigma^\square \to A_{\Sigma_L\cup\Sigma_R}^\square \to \hat{\bigotimes}_{v\in S_L\cup S_R}\mathbb{F}) = \hat{\bigotimes}_{v\in S_L\cup S_R}\mathfrak{m}_v^\square$. It follows that $\Gamma^\zeta$ is identified with

$$\ker(A_\Sigma^\square \to A_{\Sigma_p\cup\Sigma_A}^\square \hat{\otimes} (\hat{\bigotimes}_{v\in S_L\cup S_R}\mathbb{F}) \to R)$$

and the latter is generated by $\Gamma^\zeta_{\Sigma_p\cup\Sigma_A}$ and $\Gamma^\zeta_{\Sigma_L\cup\Sigma_L}$. Hence

$$A_\Sigma^{\square,\Gamma^\zeta} \simeq A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta} \otimes A_{\Sigma_L\cup\Sigma_R}^{\square,\Gamma^\zeta} \simeq A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta} \otimes A_{\Sigma_L\cup\Sigma_R}^\square$$

where, by slight abuse of notation, $A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta}$ (resp. $A_{\Sigma_L\cup\Sigma_R}^{\square,\Gamma^\zeta}$) denotes the completed localisation of $A_{\Sigma_p\cup\Sigma_A}^\square$ (resp. $A_{\Sigma_L\cup\Sigma_R}^\square$) at $\Gamma^\zeta_{\Sigma_p\cup\Sigma_A}$ (resp. $\Gamma^\zeta_{\Sigma_L\cup\Sigma_L}$). It suffices to understand $A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta}$.

For each minimal prime $\triangle$ of $\Lambda$, we observe that $(A_{\Sigma_p\cup\Sigma_A}^\square/\triangle)^{\Gamma^\zeta}$ is formally smooth over $(\Lambda/\triangle)^\Gamma$. This follows by computing the relative tangent space of $(\Lambda/\triangle)^\Gamma \to (A_{\Sigma_p\cup\Sigma_A}^\square/\triangle)^{\Gamma^\zeta}$. As a result, we may conclude that $(A_{\Sigma_p\cup\Sigma_A}^\square/\triangle)^{\Gamma^\zeta}$ is a regular local ring.

We observe Spec $\left[(A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta}/\triangle)\hat{\otimes}A_{\Sigma_L\cup\Sigma_R}\right][1/p]$ is connected, hence so is Spec $\left[A_\Sigma^{\square,\Gamma^\zeta}/\triangle\right][1/p]$; on the other hand, Spec $\left[A_\Sigma^{\square,\Gamma^\zeta}/\triangle\right][1/p]$ is regular. Combining, we see that Spec $\left[A_\Sigma^{\square,\Gamma^\zeta}/\triangle\right][1/p]$ is a domain, and the first assertion follows from this. To see that Spec $\left[A_\Sigma^{\square,\Gamma^\zeta}/\triangle\right][1/p]$ is regular, it suffices to show that Spec $\left[A_{\Sigma_p\cup\Sigma_A}^{\square,\Gamma^\zeta}/\triangle\right][1/p]$ is regular; in fact, it is enough to show Spec $\left[A_{\Sigma,\Gamma^\zeta}^\square/\triangle\right][1/p]$ because of the observation that the map

$$\left[A_{\Sigma,\Gamma^\zeta}^\square/\triangle\right][1/p] \to \left[A_\Sigma^{\square,\Gamma^\zeta}/\triangle\right][1/p]$$

is faithfully flat and regular, and Theorem 32.2 in [62]. The regularity of Spec $\left[A_{\Sigma,\Gamma^\zeta}^\square/\triangle\right][1/p]$ follows from results in Section 2.3.

Since $A_\Sigma^\square$ is Cohen-Macaulay, it follows from Theorem 2.1.3 in [13] for example that the localisation of $A_\Sigma^\square$ at $\Gamma$ is Cohen-Macaulay. Since the morphism passing from the localisation to its completion is regular, the completion $A_\Sigma^{\square,\Gamma}$ is Cohen-Macaulay.

It follows from results in Section 2.3 that $A_{\Sigma_p}^\square/(\triangle,\lambda)$ is generically reduced. By Lemma 3.3 in [8], this proves that $A_\Sigma^\square/(\triangle,\lambda)$ is generically reduced. Furthermore, it follows that the localisation of $A_\Sigma^\square$ at $\Gamma$ is generically reduced. Since it is excellent, the completion $A_\Sigma^{\square,\Gamma}$ is generically reduced.

It follows from Lemma 3.3 in [8] that every prime of the localisation of $A_\Sigma^\square$ at $\Gamma$, minimal amongst those containing $\lambda$, contains a unique minimal prime of the localisation. To pass to the completion, we make appeal to Proposition 1.6 in [89]. $\square$

*Proof of the proposition.* Let $\triangle$ denote a minimal ideal of $\Lambda$. We observe that the $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$-depth of $H^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is greater than and equal to the $(\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}/\triangle)$-depth of $H^{\square,\Gamma}_{\Sigma_Q}/\triangle$; by the freeness, the latter equals the Krull dimension of $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}/\triangle$ which is

$$\begin{aligned} & (1+[F:\mathbb{Q}]+2+\gamma_F)+4|T|+q-[F:\mathbb{Q}] \\ = \;& q+1+[F:\mathbb{Q}]+\gamma_F+4|T| \end{aligned}$$

where $T = S_p \cup S_R \cup S_L \cup S_A$.

It then follows from Lemma 27 that, when $\zeta$ is distinct, $H^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is nearly faithful over $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$. By Lemma 2.2 in [83], $H^{\square,\Gamma}_{\Sigma_Q}/(\triangle,\lambda)$ is nearly faithful over $A^{\square,\Gamma}_{\Sigma_Q}/(\triangle,\lambda)$ in the case when $\zeta$ is trivial. By Lemma 2.2 in [83] again, $H^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is nearly faithful over $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$ and therefore $H^{\square,\Gamma}_{\Sigma_Q}$ is nearly faithful over $A^{\square,\Gamma}_{\Sigma_Q}$. By Lemma 2.2 in [83], $H^{\square,\Gamma}_{\Sigma_Q}/J \simeq H^{\Gamma}_{\Sigma}$ is nearly faithful over $A^{\square,\Gamma}_{\Sigma_Q}/J \simeq A^{\square,\Gamma}_{\Sigma}$. Note that $A^{\square,\Gamma}_{\Sigma} \simeq A^{\square,\Gamma}_{\Sigma_Q}/J \simeq R^{\square,\Gamma}_{\Sigma_Q}/J \simeq R^{\Gamma}_{\Sigma}$.

On the other hand, one observes that $p$ and the generators of $J$ define a regular sequence of $A^{\square,\Gamma}_{\Sigma_Q}$. One then concludes, as in the proofs of Theorem 16 and Corollary 17 that $R^{\Gamma}_{\Sigma}[1/p] \simeq A^{\square,\Gamma}_{\Sigma}[1/p]$ is reduced. On the other hand, $R^{\square,\Gamma}_{\Sigma_Q} \simeq A^{\square,\Gamma}_{\Sigma_Q}$ is a noetherian local Cohen-Macaulay ring and $p$ is $R^{\square,\Gamma}_{\Sigma_Q}/J$-regular, $R^{\Gamma}_{\Sigma}$ is $p$-torsion free and one concludes that $R^{\Gamma}_{\Sigma}$ injects into $R^{\Gamma}_{\Sigma}[1/p]$ and therefore that $R^{\Gamma}_{\Sigma}$ is reduced. Because of this, the nearly faithfulness of $H^{\Gamma}_{\Sigma}$ over $R^{\Gamma}_{\Sigma}$ is promoted to the faithfulness. □

**Proposition 28.** *Any prime contained in an admissible prime $\Gamma$ in $R_{\Sigma}$ is pro-modular.*

*Proof.* By definition, $\Gamma$ contains $J = \ker(r : R_{\Sigma} \twoheadrightarrow T_{\Sigma})$. It suffices to show that a minimal prime $\triangle$, contained in $\Gamma$, contains $J$. By Proposition 25, $JR^{\Gamma}_{\Sigma} = 0$. Since $R_{\Sigma,\Gamma} \to R^{\Gamma}_{\Sigma}$, where $R_{\Sigma,\Gamma}$ denote the localisation of $R_{\Sigma}$ at $\Gamma$, is faithfully flat, $JR_{\Sigma,\Gamma} = 0$. It therefore follows that the ideal $JR_{\Sigma,\triangle}$ of the localisation $R_{\Sigma,\triangle}$ is 0, and $J \subset \triangle$. □

## 5.4 Finding $\Gamma$

**Lemma 29.** *Suppose that $E$ is a quadratic extension of $F$ in which not every place of $F$ above $p$ splits completely; and suppose that $\overline{\rho}$ is $E$-dihedral. If $\Gamma \subset R_{\Sigma}$ is a prime as defined at the beginning of Section 5, then the lifting $\rho_{\Gamma}$ of $\overline{\rho}$ over $R_{\Gamma}$ is not dihedral.*

*Proof.* If $\rho_{\Gamma}$ were dihedral, it would be $E$-dihedral and it would follow from Lemma 2.2.1 in [79] that every prime of $F$ above $p$ splits completely in $E$. This contradicts the assumption on $E$. □

When the quadratic extension $E$, from which $\overline{\rho}$ is induced, is totally real over $F$ (e.g. $E = F^{+} \subset F(\zeta_p)$ when $p > 2$), it is possible to allow every place of $F$ above $p$ to split completely in $E$. In fact, it is possible to ascertain that $\rho_{\Gamma}$ is not $E$-dihedral at all (even if it is still dihedral). To this end, let $E$ be a totally real quadratic extension of $F$ and let $E_S$ denote the maximal pro-$p$-extension of $E$ unramified outside the places above $S$ such that $\mathrm{Gal}(E/F)$ acts non-trivially on $\mathrm{Gal}(E_S/E)$.

Suppose that $E = (E \cap A(\mathbb{Q}))F$ where $A(\mathbb{Q})$ is the maximal abelian extension of $\mathbb{Q}$.

**Lemma 30.** *The $\mathbb{Z}_p$-rank* $\mathrm{rk}\,\mathrm{Gal}(E_S/E)$ *of* $\mathrm{Gal}(E_S/E)$ *satisfies* $\mathrm{rk}\,\mathrm{Gal}(E_S/E) \le [F:\mathbb{Q}] - [F \cap A(\mathbb{Q}):\mathbb{Q}]$.

*Proof.* By assumption, $E \cap A(\mathbb{Q})$ is a abelian, totally real quadratic extension of $F \cap A(\mathbb{Q})$. Hence the $\mathbb{Z}$-rank of $(E \cap A(\mathbb{Q}))^\times$ is $2[F \cap A(\mathbb{Q}) : \mathbb{Q}] - 1$, and the subgroup $\Gamma$ of the units $(E \cap A(\mathbb{Q}))^\times$ on which $\mathrm{Gal}((E \cap A(\mathbb{Q}))/(F \cap A(\mathbb{Q})))$ acts non-trivially has $\mathbb{Z}$-rank $2[F \cap A(\mathbb{Q}) : \mathbb{Q}] - 1 - ([F \cap A(\mathbb{Q}) : \mathbb{Q}] - 1) = [F \cap A(\mathbb{Q}) : \mathbb{Q}]$. It follows from the Leopoldt 'conjecture' for abelian extensions of $\mathbb{Q}$ that the closure $\overline{\Gamma}$ of $\Gamma$ in the $p$-adic completion $\overline{\mathscr{O}}_E^\times$ of $\mathscr{O}_E^\times$ where $\mathrm{Gal}(L/F)$ acts non-trivially, has rank at least $[F \cap A(\mathbb{Q}) : \mathbb{Q}]$. We then deduce that $\mathrm{rk}\, \mathrm{Gal}(E_S/E) = \mathrm{rk}\, \overline{\mathscr{O}}_E^\times/\overline{\Gamma} \leq [F : \mathbb{Q}] - [F \cap A(\mathbb{Q})]$. □

**Lemma 31.** *If $I$ be an ideal of $R_\Sigma$ such that*

- *the determinant of $\rho_I : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R_\Sigma) \to \mathrm{GL}_2(R_\Sigma/I)$ is of finite order,*
- *the Krull dimension $\dim R_\Sigma/I > ([F : \mathbb{Q}] - [F \cap A(\mathbb{Q}) : \mathbb{Q}]) + 1$,*

*then $\rho_I$ is not $E$-dihedral for any totally real quadratic extension $E$ of $F$ satisfying $E = (E \cap A(\mathbb{Q}))F$.*

*Proof.* If $\rho_I$, whose determinant is of finite order, were $E$-dihedral, then $\overline{\rho}$ would be $E$-dihedral and we might think of $\mathscr{O}[[\mathrm{Gal}(E_S/E)]]$ as the universal ring for $E$-dihedral deformations of the $E$-dihedral $\overline{\rho}$ whose determinant equals the Teichmuller lift of $\det \overline{\rho}$; the natural quotient $R_\Sigma \to R_\Sigma/I$ would factor as a composite $R_\Sigma \to \mathscr{O}[[\mathrm{Gal}(E_S/E)]] \to R_\Sigma/I$ of surjections, but this is impossible as $\dim R_\Sigma/I > ([F : \mathbb{Q}] - [F \cap A(\mathbb{Q}) : \mathbb{Q}]) + 1 \geq \mathrm{rk}\, \mathrm{Gal}(E_S/E) + 1$. □

**Proposition 32.** *Suppose that $F$ satisfies the following conditions:*

- *$[F_\mathrm{v} : \mathbb{Q}_p] > 4|S_R|$ for every place $\mathrm{v}$ of $F$ above $p$,*
- *the degree $[F \cap A(\mathbb{Q}) : \mathbb{Q}]$ of the maximal abelian subextension of $F$ over $\mathbb{Q}$ is strictly greater than $4|S_R|$.*

*Then $R_\Sigma$ contains an admissible prime.*

*Proof.* We make appeal to Lemma 1.9 in [89]. The determinant of the universal Galois representation $\rho_\Sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R_\Sigma)$ defines a map $\Lambda^p \to \Lambda \to R_\Sigma$ and let $\Delta_\Sigma$ denote the ideal of $R_\Sigma$ generated by the image.

Let $S$ be the quotient $R_\Sigma/(\lambda, J, \Delta_\Sigma)$, where $J$ denotes the kernel of $R_\Sigma \to T_\Sigma$. Let $X_\Sigma$ denote a family of countably many ideals in $R_\Sigma$. If there exists a non-negative integer $N$ such that $\dim S \geq N$ and $\dim S/I_S < N$ holds for the image $I_S$ of every ideal $I$ in $X_\Sigma$, then it follows that there exists a co-height one prime $\Gamma$ of $R_\Sigma$ that does *not* contain any $I$ in $X_\Sigma$. Indeed a such prime will satisfies all the conditions for it to be admissible except that $\rho_\Gamma$ is trivial at the places in $S_R$.

We define $N = [F : \mathbb{Q}] - 4|S_R|$ (this is positive by the first assumption). For $X_\Sigma$, we choose

- the image of the ideal $I_\mathrm{v}$ of $\Lambda$, for every place $\mathrm{v}$ above $p$, corresponding to the subset of two identical characters of $\Delta_\mathrm{v}$; by assumption,

  $$\dim S/I_\mathrm{v} \leq (\dim \Lambda - (1 + 1 + \gamma_F)) - [F_\mathrm{v} : \mathbb{Q}_p] = [F : \mathbb{Q}] - [F_\mathrm{v} : \mathbb{Q}_p] < N$$

  holds,

- the ideal $I = \ker(R_\Sigma \to \mathscr{O}[[\mathrm{Gal}(E_S/E)]])$ when $\overline{\rho}$ is $E$-dihedral for a totally real quadratic extension $E$ over $F$ (in which every place of $F$ above $p$ may, or may not, split completely in $E$); by assumption and Lemma 30,

$$\dim S/I \leq \dim \mathbb{F}[[\mathrm{Gal}(E_S/E)]] \leq [F : \mathbb{Q}] - [F \cap A(\mathbb{Q}) : \mathbb{Q}] < N$$

holds.

Let $\mathfrak{m}_v^\zeta \subset R_\Sigma$, for every place v in $S_R$, denotes the image of the maximal ideal $\mathfrak{m}_v^{\square,\zeta}$ of $R_v^\square/I_v^{\square,\zeta}$ under $R_v^\square/I_v^{\square,\zeta} \to R_\Sigma^\square \to R_\Sigma$. Let $I$ denote the ideal of $R_\Sigma$ generated by $\{\mathfrak{m}_v^\zeta\}$. It then follows from Theorem 15.1 in [62] that $\dim S/I_S \geq \dim S - 4|S_R| \geq [F : \mathbb{Q}] - 4|S_R| = N$. Lemma 1.9 in [89] then finds an admissible prime. □

Finally, we prove that, given the assumption on $F$ at the beginning of the section, there cannot possibly be a component of $R_\Sigma$ (i.e., a minimal ideal) that is *not* pro-modular.

**Corollary 33.** *Let $F$ be as assumed in the proposition. Suppose that $|S_R| > 1$. Every prime of $R_\Sigma$ is pro-modular.*

*Proof.* Let $A$ (resp. $A^\neg$) denote a set of minimal primes of $R_\Sigma$ which are pro-modular (resp. not pro-modular), and suppose furthermore that $A$ and $A^\neg$ are disjoint and their union equals all the minimal primes of $R_\Sigma$. We know $A$ is not empty by the existence of a admissible prime above, and it suffices to prove that $A^\neg$ is empty. Suppose that $A^\neg$ is not empty. It would then follows that there exist $\triangle$ in $A$, and $\triangle^\neg$ in $A^\neg$ such that

$$\dim R_\Sigma/(\triangle, \triangle^\neg) \geq c(R_\Sigma) \geq [F : \mathbb{Q}] + \gamma_F - 2|S_R| - 1.$$

Since $\triangle$ is pro-modular, it contains $J$. As a result, $R_\Sigma/(\triangle, \triangle^\neg, \lambda, \Delta_\Sigma)$ is a quotient of $R_\Sigma/(\lambda, J, \Delta_\Sigma)$. Since

$$\dim R_\Sigma/(\triangle, \triangle^\neg, \lambda, \Delta_\Sigma) \geq [F : \mathbb{Q}] + \gamma_F - 2|S_R| - 1 - (1 + 1 + \gamma_F) = [F : \mathbb{Q}] - 2|S_R| - 3 \geq N$$

for $N$ in the proof of Proposition 32, it then follows that there would be an admissible prime of $R_\Sigma$ containing, in particular, $\triangle^\neg$. By Proposition 28, $\triangle^\neg$ would then be pro-modular, and this contradicts the assumption about $\triangle^\neg$. □

## 5.5 A quick reminder about pseudo-representation theory

A pseudo-representation $D : E \to R$ (over $R$) of dimension $r$ is a polynomial law of degree $r$.

A pseudo-representation $D : E \to R$ is said to be of Cayley-Hamilton type (CH-type for short) if every element of $E$ satisfies the characteristic polynomial of $D$. Let $T(D)$ denote the trace of $D$.

A pseudo-representation $D : E \to R$ is said to be of Azumaya type (A-type for short) if there exists a projective $R$ of finite rank $V$ over $R$ such that it factors the pseudo-representation $\det : \mathrm{End}(V) \to R$. If a pseudu-representation is A-type, it is of CH type.

A pseudo-representation $D : E \to R$ is said to be a pseudo-representation of $\Gamma$ over $R$ if $E = R[\Gamma]$. A pseudo-representation $R[\Gamma] \to R$ of $\Gamma$ over $R$ is said to be of CH-type (resp. A-type) if there exist

- a finitely generated $R$-module $E$ (resp. $\mathrm{End}(V)$ for a projective $R$-module of finite rank $V$) which comes equipped with a pseudo-representation $D : E \to R$ (resp. $\det : \mathrm{End}(V) \to R$)
- $\rho : \Gamma \to E^\times$ which gives rise to $\rho : R[\Gamma] \to E$

such that $R[G] \to R$ is given by

$$R[\Gamma] \xrightarrow{\rho} E \xrightarrow{D} R$$

$$(\text{resp. } R[\Gamma] \xrightarrow{\rho} \mathrm{End}(V) \xrightarrow{\det} R)$$

An algebra $E$ over $R$ is a generalised matrix algebra (GMA for short) if it comes equipped with a data of idempotents, or a GMA-structure over $R$, as defined in [9]. If $E$ is a GMA, the GMA -structure defines a trace function we shall denote by $T(E) : E \to R$. The following is stated as Lemma 3.1.3 of [91]:

**Proposition 34.** *Given a GMA $E$ over $R$, there is a CH pseudo-representation $D = D(E) : E \to R$ with trace $T(D) = T(E)$.*

Given a GMA-algebra $E$ over $R$ of type $(1, 1)$, there is an isomorphism of $R$-modules $E \simeq \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \simeq R$, $D \simeq R$ and $B$ and $C$ are finite $R$-algebras.

If $(D : E \to R, \rho : \Gamma \to E^\times)$ is a CH pseudo-representation of $\Gamma$ over $R$ and if $E$ is a GMA of type $(1, 1)$ over $R$, then pre-composing with $\rho : R[\Gamma] \to E$ defines a psedo-representation $(A : \Gamma \to R, D : \Gamma \to R, T = A + D : \Gamma \to R, X = BC : \Gamma \times \Gamma \to R)$ as defined in [96].

Let $D_{\mathbb{F}} : \mathbb{F}[\Gamma] \to \mathbb{F}$ be a pseudo-representation of $\Gamma$ over $\mathbb{F}$. A pseudo-representation $(D : E \to R, \rho : \Gamma \to E^\times)$ over a local ring $R$ with residue field $\mathbb{F}$ is a pseudo-deformation of $D_{\mathbb{F}}$ if $D \otimes_R \mathbb{F} \simeq D_{\mathbb{F}}$.

**Proposition 35.** *Let $\overline{\rho} : \Gamma \to \mathrm{GL}(V_{\mathbb{F}}) \simeq \mathrm{GL}_2(\mathbb{F})$ be a representation of $\Gamma$ over $\mathbb{F}$. Suppose that $\overline{\rho}$ is multiplicity-free and let $D_{\mathbb{F}}(\overline{\rho}) : \mathbb{F}[\Gamma] \to \mathbb{F}$ denote the associated pseudo-representation of $\Gamma$ over $\mathbb{F}$. If $(D : E \to R, \rho : \Gamma \to E^\times)$ is a CH pseudo-deformation of $D_{\mathbb{F}}(\overline{\rho})$ over a noetherian local Henselian ring $R$ with residue field $\mathbb{F}$, then $E$ is a GMA algebra over $R$ and $D(E) = D$.*

*Proof.* This is stated as Theorem 3.2.2 in [91]. □

As before, let $\hat{\mathscr{C}}$ be the category of complete noetherian local $\mathscr{O}$-algebras with residue field isomorphic to $\mathbb{F}$. Let $S = S^{\overline{\rho}}$ denote the universal ring for pseudo-deformations $R[\Gamma] \to R$ of $D_{\mathbb{F}}(\overline{\rho})$ over $\hat{\mathscr{C}}$. There exists (Theorem 2.2.9 in [91]) a universal CH-pseudo-deformations $(D_S : E_S \to S, \rho_S : \Gamma \to E_S^\times)$ of $D_{\mathbb{F}}(\overline{\rho})$ over $S$. The CH-algebra $E_S$ is given as the 'maximal CH quotient of $S[\Gamma]$ and factors the universal pseudo-deformation $S[\Gamma] \to S$ of $D_{\mathbb{F}}(\overline{\rho})$ as

$$S[\Gamma] \xrightarrow{\rho_S} E_S \xrightarrow{D_S} S.$$

The local conditions prescribed by a deformation data $\Sigma$ single out CH-pseudo-deformations satisfying conditions (Section 2.3 in [91]), which we shall refer to as CH pseudo-deformations *of type* $\Sigma$, and there exists a universal CH-pseudo-deformations of $D_{\mathbb{F}}(\overline{\rho})$ of type $\Sigma$ over the quotient $S_\Sigma^{\overline{\rho}}$ of $S^{\overline{\rho}}$ (Theorem 2.5.3 in [91]).

We define a pseudo-deformation $D : R[\Gamma] \to R$ of $D_{\mathbb{F}}(\overline{\rho})$ to be of type $\Sigma$ if the CH-module $E_S \otimes_S R$ over $R$ is of type $\Sigma$. In this optic, the complete noetherian local ring $S_\Sigma^{\overline{\rho}}$ represents (Theorem 2.5.5 in [91]) the pseudo-deformations of $D_{\mathbb{F}}(\overline{\rho})$ of type $\Sigma$.

### 5.6 $\overline{\rho}$ is induced from a character of a CM field in which every place of $F$ above $p$ splits completely

We now suppose that $F$ satisfies the following conditions (whether $p > 2$ or $p = 2$):

- $F$ is even,
- $\overline{\rho}$ is irreducible,
- $\overline{\rho}$ is abelian when restricted to $\mathrm{Gal}(\overline{F}/E)$ for a quadratic imaginary extension $E$ of $F$ in which every place v of $F$ above $p$ splits completely.

We may suppose that $\overline{\rho}$ is induced from a character $\overline{\zeta} : \mathrm{Gal}(\overline{F}/E) \to \mathbb{F}^{\times}$. In particular, the restriction of $\overline{\rho}$ to $\mathrm{Gal}(F_S/E)$ is the direct sum of $\overline{\zeta}$ and its conjugate character $\overline{\zeta}_c$.

Suppose that the image of $\overline{\rho}$ is a dihedral group $D_{2n}$ of order $2n$. An abelian subgroup of $D_{2n}$ of index 2 is either a cyclic subgroup of order $n$, or one of the two dihedral groups of order $n$. It therefore follows that, unless, $n = 2$ or 4, there is a unique index 2 abelian subgroup, a cyclic group $C_n$ of order $n$ generated by the 'roration'.

In the case of $D_8$ (resp. $D_4$), it is either $C_4$ or one of the two dihedral groups $D_2 \simeq C_2 \times C_2$ of order 4 (resp. one of the three abelian group isomorphic to $C_2$). However, by Dickson's classification of subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ (Theorem 2.47 in [25] for example), the image of $\overline{\rho}$ can be $D_8$ or $D_4$, only when $p > 2$.

In conclusion, unless $p > 2$ and the image of $\overline{\rho}$ is isomorphic to $D_4$ or $D_8$, the quadratic (imaginary) field extension $E$ from which $\overline{\rho}$ is induced is unique.

We now recall a theory of CM forms in a manner similar to the one in Section 3.

Let $F^+$ be an imaginary quadratic extension in which every place of $F$ above $p$ splits completely. Let $\Delta_S$ denote the pro-$p$ completion of the Galois group of the maximal abelian *pro-$p$* extension of $F^+$ unramified outside the set of places in $F^+$ lying above those in $S$ that do not ramify in $F^+$. Every place v of $F$ above $p$ is assumed to split completely in $F^+$ and we *choose* one of the two places of $F^+$ above v. This defines an injection of $\Lambda$ into the group algebra $\Lambda_S = \mathscr{O}[[\Delta_S]]$ of $\mathbb{Z}_p$-rank $1 + [F : \mathbb{Q}] + \gamma_F$.

There is a 'universal' character

$$\mathrm{Gal}(\overline{F}/F^+) \to \Delta_S \hookrightarrow \Lambda_S^{\times},$$

unramified outside the set of places in $F^+$ lying above those in $S$ that do not ramify in $F^+$, and its induction to $\mathrm{Gal}(\overline{F}/F)$ defines

$$\rho_S : \mathrm{Gal}(\overline{F}/F) \to \Delta_S \hookrightarrow \mathrm{GL}_2(\Lambda_S)$$

unramified outside $S$ and locally split at every place of $F$ above $p$.

As in Section 3, for an open compact subgroup $U$ of $G(\mathbb{A}^{\infty})$, let $S(U, L/\mathscr{O})$ denote the $\mathscr{O}$-module of cusp forms (defined on a totally definite quaternion algebra over $F$) of trivial weight and level $U$ on $G(\mathbb{A})$. Let $U^{[r]}$ be an open compact subgroup of $G(\mathbb{A})$ which is maximal compact hyperspecial at v outside $S$ and reduces mod $r$-th power of $\pi_v$ to the upper-triangular unipotent matrices at v in $S_p$. Let $eS(U, L/\mathscr{O})$ denote the direct limit of $eS(U^{[r]}, L/\mathscr{O})$ over $r$. Let $C(\Delta_S, L/\mathscr{O})$ denote the space of continuous functions on $\Delta_S$ with values in $L/\mathscr{O}$.

Hida proves that the construction $\phi \mapsto \theta(\phi)$ associating a Hecke character $\phi : F^{+\times}\backslash\mathbb{A}^\times_{F^+} \to \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ to the ($q$-expansion of a) $\theta$-series $\theta(\phi)$ can be $p$-adically interpolated to a correspondence

$$\tau : C(\Delta_S, L/\mathscr{O}) \to eS(U, L/\mathscr{O})$$

associating an element $\phi$ in $C(\Delta_S, L/\mathscr{O})$ to a $p$-ordinary normalised eigenform $\theta(\phi)$, where $U$ is defined such that if v is a finite place dividing either the relative conductor of $\phi$ in $F$ or the relative discriminant of $F^+$ over $F$, then $U_v \subset \mathrm{GL}_2(\mathscr{O}_{F_v})$ is the pre-image, by $\mathrm{GL}_2(\mathscr{O}_{F_v}) \to \mathrm{GL}_2(\mathbb{F}_v)$ of the subgroup of upper triangular matrices.

Let $S(\tau, U, L/\mathscr{O})$ denote the image of $\tau$ in $eS(U, L/\mathscr{O})$, and let $S(\tau, U, \mathscr{O})$ denote the Pontryagin dual of $S(\tau, U, L/\mathscr{O})$. Let $T(\tau, U, \mathscr{O}) \subset \mathrm{End}(S(\tau, U, \mathscr{O}))$ denote the corresponding $p$-ordinary Hecke algebra as defined in Section 3. This is naturally a $\Lambda_S$-algebra, and let

$$T_{S_\tau} \subset \mathrm{End}(S(\tau, U, \mathscr{O})_{\mathfrak{m}})$$

denote the localisation at the maximal ideal $\mathfrak{m}$ corresponding to $\overline{\rho}$. It follows that there exists a $F^+$-dihedral representation

$$\rho_{S_\tau} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(T_{\Sigma_\tau})$$

such that $\rho_{S_\tau}$ is split at v in $S_p$; and there is a $\Lambda_S$-algebra homomorphism, 'a $\Lambda_S$-adic form',

$$T_{S_\tau} \to \Lambda_S$$

sending $T_v$ (resp. $S_v$) to $\mathrm{tr}\,\rho_S(\mathrm{Frob}_v)$ (resp. $(\mathbf{N}_{F/\mathbb{Q}}\mathrm{v})^{-1}\det\rho_S(\mathrm{Frob}_v)$) for every place v not lying in $S$. By construction, this is an isomorphism.

If an imaginary quadratic extension $F^+$ as above has relative discriminant $D$ defined as the product of places in a subset $S^D$ of (distinct) places in $S_R \cup S_L$ and $\phi$ is a character of $F^+$ of conductor a product of (distinct) places in $(S_R \cup S_L) - S^D$, then $\theta(\phi)$ generates (via the Jacquet-Langlands correspondence) the subspace $eS_\Sigma(\tau, U_\Sigma, \mathscr{O})$ of cusp forms with complex multiplication by $F^+$ in $eS_\Sigma(U_\Sigma, \mathscr{O})$ as defined in Section 3 (whether $\zeta$ is trivial or not). The correspoding Hecke algebra $T_{\Sigma_\tau} \subset \mathrm{End}(eS_\Sigma(\tau, U_\Sigma, \mathscr{O})_{\mathfrak{m}})$ defines an 'irreducible component' Spec $T_{\Sigma_\tau}$ of Spec $T_\Sigma$.

In terms of Galois representations, we have

$$T_{S_\tau} \otimes_{\Lambda_S} R_\Sigma \simeq T_{\Sigma_\tau}.$$

**Proposition 36.** *Suppose that $F$ satisfies the first condition in Proposition 32. Let $\rho_\Sigma$ denote the universal deformation of $\overline{\rho}$ over $R_\Sigma$.*

- *If $\rho_\Sigma$ is not dihedral, then there exists an admissible prime $\Gamma$ of $R_\Sigma$ such that the surjection $R_\Sigma \to T_\Sigma$ gives rise to an isomorphism*

$$R^\Gamma_\Sigma \simeq T^\Gamma_\Sigma.$$

- *If $\rho_\Sigma$ is dihedral, then $R_\Sigma$ is pro-modular.*

- *Every prime of $R_\Sigma$ is pro-modular.*

*Proof.* Suppose that $\rho_\Sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R_\Sigma)$ is not dihedral. In particular, $\rho$ is not abelian when restricted to any one $\Delta$ of the possible abelian subgroups of index 2 in the image of $\overline{\rho}$– as discussed at the beginning of Section 5.6, $\Delta$ is unique ($= \mathrm{Gal}(\overline{F}/E)$), unless $p > 2$ and the image of $\overline{\rho}$ is isomorphic to $D_4$ or $D_8$ in which case there are three possible index subgroups).

By assumption, there exists a $p$-ordinary Hilbert modular eigenform $\Pi$ whose associated Galois representation $\rho_\Pi$ is of type $\Sigma$ and defines a deformation of $\overline{\rho}$. Raising the level at a finite place v of $F$ with $\mathbf{N}_{F/\mathbb{Q}} \equiv 1 \bmod p$ at which $\overline{\rho}$ is trivial (e.g. a place in $S_R$ prescribed in $\Sigma$) if necessary, we may assume furthermore that $\rho_\Pi$ is not dihedral of type $\Sigma$. This is possible because of the observation that the corresponding Hecke modules (denoted earlier by $H_\Sigma$), in the case where '$\zeta$' is distinct and in the case where it is trivial, are congruent, and of the Deligne-Serre's Lemma 6.11 in [32]. Granted, there is a $\Lambda$-adic form

$$T_\Sigma \to R$$

passing through $\Pi$, over the integral closure $R$ (whose dimension equal to $\dim T_\Sigma$) of a finite extension of the field of fractions of $\Lambda$. In particular, its associated representation $\rho = \rho_\triangle : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R))$ is not dihedral.

Following Section 5.5, $\rho$ gives rise to a free-module $V$ of rank 2 over $R$ such that ($D = \det : \mathrm{End}(V) \to R, \rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{End}(V)^\times = \mathrm{GL}(V)$) defines a pseudo-deformation of the pseudo-representation $D_{\mathbb{F}}(\overline{\rho})$ of $\mathrm{Gal}(\overline{F}/F)$ over $\mathbb{F}$. It follows from Proposition 35 that $\mathrm{End}(V)$ is isomorphic to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some finitely generated $R$-modules $A$, $B$, $C$ and $D$, and $A$ and $D$ are both isomorphic to $R$; and we may write $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V)$ as

$$\sigma \mapsto \begin{pmatrix} A(\sigma) & B(\sigma) \\ C(\sigma) & D(\sigma) \end{pmatrix}$$

and $X(\sigma, \tau) = B(\sigma)C(\tau)$. Since $\rho$ is irreducible, the map $\mathrm{Gal}(\overline{F}/F)$ acts non-trivially on $B$ and $C$. It follows from the assumption– $\rho$ is not dihedral– that for any possible index 2 abelian subgroup of $\mathrm{Gal}(\overline{F}/F)$, which we shall again denote by $\Delta$ by slight abuse of notation, the induced map on $\Delta$ also acts non-trivial on $B$ and $C$ simultaneously (since $\Delta$ is a normal subgroup of $\mathrm{Gal}(\overline{F}/F)$). Fixing $\Delta$, it follows that there exist $r$ and $s$ in $\Delta$ such that $\mathfrak{L} = X(r, s)$ is non-zero in $R$; and, for every element $\sigma$ in $\Delta$,

$$\sigma \mapsto \begin{pmatrix} A(\sigma) & X(\sigma, s)/\mathfrak{L} \\ X(r, \sigma) & D(\sigma) \end{pmatrix}$$

defines an irreducible (in particular, non-abelian) representation of $\Delta$ over $R^{\mathfrak{L}} = R[1/\mathfrak{L}]$. Let $B^{\mathfrak{L}} = B^{\mathfrak{L}}_\Delta : \Delta \to R^{\mathfrak{L}}$ denote the non-zero map sending $\sigma$ to $X(\sigma, s)/\mathfrak{L}$. It follows, since $R$ is a noetherian domain, that there is a height one prime $\Gamma^{\mathfrak{L}}$ of $R^{\mathfrak{L}}$ that does not contain the image of $B^{\mathfrak{L}}$. Unless $\dim R^{\mathfrak{L}} \leq 1$, there are infinitely many height one primes in $R^{\mathfrak{L}}$, and only finitely many of them contain the image of $B^{\mathfrak{L}}$. It therefore follows that one can find a such height one prime $\Gamma^{\mathfrak{L}}$ in $R^{\mathfrak{L}}$ that does not contain the image of $B^{\mathfrak{L}}_\Delta$ for every possible $\Delta$. Let $S^{\mathfrak{L}}$ denote $R^{\mathfrak{L}}/\Gamma^{\mathfrak{L}}$; the induced map $B^{\mathfrak{L}} : \Delta \to S^{\mathfrak{L}}$ remains non-zero (for every $\Delta$). Since $\dim S^{\mathfrak{L}} \leq \dim R^{\mathfrak{L}} - 1 = \dim R - 1$ (see [62], p.30), we may repeat the process to assume that $S^{\mathfrak{L}}$ is a Dedekind domain (by replacing it by the integral closure in its field of fractions), or indeed a DVR (by localising it further at a generator of the ideal of $R^{\mathfrak{L}}$ where $B^{\mathfrak{L}}$ vanishes).

As in the proof of Lemma 2.13 in [80], it is then possible to construct a non-zero cocycle $\overline{B}$ : $\mathrm{Gal}(\overline{F}/F) \to \mathbb{F}$ which remains non-zero when restricted to any index 2 subgroup $\Delta$. Since $\overline{\rho}$ is (absolutely) irreducible, the centraliser of $\overline{\rho}$ is $\mathbb{F}$; and it follows that if $\overline{B}$ were a coboundary, it would be (a scalar multiple of) the trivial cocycle by conjugation. Since $\overline{B}$ is not trivial by definition, it is not a coboundary. Corresponding to the cocyle, there exists an infinitesimal deformation

$$\overline{\rho}_{\mathbb{F}[\epsilon]} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$$

of $\overline{\rho}$ of type $\Sigma$ over the ring $\mathbb{F}[\epsilon]$ of dual numbers over $\mathbb{F}$ which is not dihedral– in particular, for every possible index 2 subgroup $\Delta$, its restriction to $\Delta$ is a non-trivial extension of the pair of conjugate constituents in the restriction of $\overline{\rho}$ at $\Delta$.

The argument in the proof of Proposition 32 now works verbatim, with the kernel of $R_\Sigma \to T_\Sigma \to \mathbb{F}[\epsilon]$ corresponding to $\overline{\rho}_{\mathbb{F}[\epsilon]}$ in place of $R_\Sigma$, to find a co-height one prime $\Gamma$ in $R_\Sigma$ whose corresponding representation $\rho_\Gamma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(R/\Gamma)$ specialises to $\overline{\rho}_{\mathbb{F}[\epsilon]}$. Since $\overline{\rho}_{\mathbb{F}[\epsilon]}$ is not dihedral, one concludes immediately that $\rho_\Gamma$ is not dihedral. The isomorphism now follows from Theorem 25.

Suppose that $\rho_\Sigma$ is $F^+$-dihedral for some quadratic extension $F^+$ of $F$. By Lemma 31 and $\dim R_\Sigma/\Delta_\Sigma \geq \dim T_\Sigma/\Delta_\Sigma \geq \dim \Lambda/\Delta_\Sigma = \dim \Lambda_p = 1 + [F : \mathbb{Q}]$, we may assume $F^+$ is CM, and furthermore assume, by the reducibility of $\rho_\Sigma$ every place v of $F$ that v splits completely in $F^+$. It then follows that there exists a 'specialisation' homomorphism $\Lambda_S \to R_\Sigma$. By composition, we have $T_{\Sigma_\tau} \to \Lambda_S \to R_\Sigma$ proving the pro-modularity. Indeed, it follows that $T_{\Sigma_\tau} \simeq R_\Sigma$ in this case.

An argument similar to the one in Corollary 33 proves that every prime of $R_\Sigma$ is pro-modular. □

**Remark**. Let $K$ denote the fixed field of $\overline{F}$ by the kernel of $\mathrm{ad}\,\overline{\rho}$. Let $G_S$ denote the maximal abelian quotient of the Galois group of the maximal extension $K_S$ of $K$ unramified outside $S$ and let $\Gamma_S = G_S/G_S^p$. Via the exact sequence

$$0 \to \mathrm{Gal}(K_S/K) \to \mathrm{Gal}(K_S/F) \to \mathrm{Gal}(K/F) \to 0,$$

the $\mathbb{F}_p$-vector space $\Gamma_S$ comes equipped with action of $\mathrm{Gal}(K/F)$ by conjugation. Let $I(\overline{\chi}_c)$ denote the representation of $\mathrm{Gal}(F_S/F)$ given by the induction of the character $\overline{\zeta}/\overline{\zeta}_c : \mathrm{Gal}(F_S/E) \to \mathbb{F}^\times$. The work [34] proves that if $\mathrm{Hom}_{\mathbb{F}[\mathrm{Gal}(K/F)]}(\Gamma_S, I(\overline{\chi}_c))$ is non-zero (resp. zero), then $\rho_\Sigma$ is non-dihedral (resp. dihedral); and that unless $\overline{\rho}$ is totally odd (which occurs only when $p = 2$ in our setting), $\mathrm{Hom}_{\mathbb{F}[\mathrm{Gal}(K/F)]}(\Gamma_S, I(\overline{\chi}_c))$ is non-zero (see Remark 3.7 and Remark 3.12 in [34]). As remarked in Remark 3.19 in [34], the work [19] also an alternative approach (albeit in a more specific setting, e.g. $F = \mathbb{Q}$) to the problem of characterising exactly when $p$-ordinary deformation rings are dihedral or not.

# 6 $\overline{\rho}$ is reducible and non-trivial

## 6.1 Pseudo-deformation of type $\Sigma$

Let $S$ be a finite set of places of $F$. Suppose that it is a disjoint union of sets $S_p$, $S_R$, $S_L$, $S_A$, $S_\infty$ as earlier defined. Suppose that

- $\overline{\chi}$ is unramified outside $S$,

- $\overline{\chi}$ is ramified at every infinite place of $F$, i.e. $\overline{\rho}$ is totally odd,
- $\overline{\chi}$ is trivial at $S_p \cup S_R \cup S_L$,
- $\overline{\chi}$ is unramified at v in $S_A$ and $\overline{\chi}(\mathrm{Frob}_v)$ is trivial (resp. non-trivial) if $p > 2$ (resp. $p = 2$).

As a result of these conditions, $\overline{\chi}$ is, in particular, non-trivial.

Let $\Gamma = \mathrm{Gal}(F_S/F)$. Let $\Sigma_{Q,N}$ denote the deformation data $(S \cup S_{Q,N}, T, \{L_v\})$, where $S_{Q,N}$ is a set of places v of $F$ such that $\mathbf{N}_{F/\mathbb{Q}}(v) \equiv 1 \bmod p^N$ at which $\overline{\chi}$ is unramified.

A class $\mathcal{D}$ in $\mathrm{Ext}^1_{\mathbb{F}[\Gamma]}(\overline{\chi}, 1)$ gives rise to a totally odd representation $\overline{\rho}_{\mathcal{D}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ unramified outside $S$, whose semi-simplification is $\overline{\rho} = 1 \oplus \overline{\chi}$.

Let $D_{\mathbb{F}}(\overline{\rho})$ denote the pseudo-representation $\mathbb{F}[\Gamma] \to \mathbb{F}$ associated to the representation $\overline{\rho} = 1 \oplus \overline{\chi} : \Gamma \to \mathrm{GL}_2(\mathbb{F})$.

Let $P_\Sigma$ denote the universal ring for pseudo-deformations of $D_{\mathbb{F}}(\overline{\rho})$ of type $\Sigma$ as defined in Section 5.5.

## 6.2 Modular pseudo-deformations

We follow the notation of Section 3. Let $\mathfrak{m}_{Q,N}$ be a maximal ideal of $eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ containing $T_v - (1 + \overline{\chi}(\mathrm{Frob}_v))$ for evert v not lying in $S$ and $U_v - 1$ for v in $S_p$, which gives rise to a pseudo-representation

$$D_{\mathbb{F}}(\overline{\rho})_{Q,N} : \mathbb{F}[\Gamma] \to eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})/\mathfrak{m}_{Q,N} \simeq \mathbb{F}$$

of GMA-type.

**Lemma 37.** *Then there exists a pseudo-deformation*

$$D_{Q,N} : \left(eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}\right)[\Gamma] \to eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}$$

*of type* $\Sigma$ *such that* $T(D_{Q,N}) = T_v$ *for every* v *not lying in* $S$.

As in Section 3, if $T_{\Sigma_{Q,N}}$ denotes the image of $eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})_{\mathfrak{m}_{Q,N}}$ in $H_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$, then the lemma gives rise to a homomorphism

$$P_{\Sigma_{Q,N}} \to T_{\Sigma_{Q,N}}.$$

## 6.3 Reducible subspaces and irreducible coverings

Let $P_{\Sigma,\Delta}$ denote the maximal reducible quotient of $P_\Sigma$. This is characterised by the property that if $D : R[\Gamma] \to R$ is a pseudo-deformation of $D_{\mathbb{F}}(\overline{\rho})$ of type $\Sigma$, then it gives rise to a unique map $P_\Sigma \to R$; and this map factors through $P_\Sigma \to P_{\Sigma,\Delta}$ if and only if $D$ is reducible.

Since $\overline{\chi}$ is non-trivial, $\overline{\rho}$ is multiplicity-free. It therefore follows that the universal CH-module $E_\Sigma$ over $P_\Sigma$ is GMA of type $(1,1)$; and we may assume $\Gamma \to E_\Sigma^\times$ to be of the form $\sigma \mapsto \begin{pmatrix} A(\sigma) & B(\sigma) \\ C(\sigma) & D(\sigma) \end{pmatrix}$ where $A(\sigma)$ (resp. $D(\sigma)$) reduces, modulo the maximal ideal of $P_\Sigma$, to 1 (resp. $\overline{\chi}$). The ideal $\ker(P_\Sigma \to P_{\Sigma,\Delta})$ is generated by $X(\sigma,\tau) = B(\sigma)C(\tau)$ as $\sigma$ and $\tau$ generated over $\Gamma$. Since $P_\Sigma$ is noetherian, $\ker(P_\Sigma \to P_{\Sigma,\Delta})$ is generated by finitely many elements $\{\mathfrak{L}\}$; and $\{\mathrm{Spec}\, P_\Sigma[\mathfrak{L}^{-1}]\}$

defines an open covering of the complement (the 'irreducible locus') Spec $P_\Sigma$ − Spec $P_{\Sigma,\Delta}$. For each $\mathfrak{L}$, there exist $r, s$ in $\Gamma$ such that $X(r,s) = \mathfrak{L}$; one easily checks that the map

$$\sigma \mapsto \begin{pmatrix} A(\sigma) & X(\sigma,s)/\mathfrak{L} \\ X(r,\sigma) & D(\sigma) \end{pmatrix}$$

defines a homomorphism, i.e. a representation, over $P_\Sigma[\mathfrak{L}^{-1}]$ with $X(r,\sigma)X(\sigma,s)/\mathfrak{L} = X(r,s) = \mathfrak{L}$. It therefore follows that $E_\Sigma \otimes_{P_\Sigma} P_\Sigma[\mathfrak{L}^{-1}]$ is of Azumaya-type. As in the proof of Proposition 36, knowing that $\sigma \mapsto X(\sigma,s)/\mathfrak{L}$ is non-zero, one may conclude that there is a prime in $P_\Sigma[\mathfrak{L}^{-1}]$ (with a DVR quotient) for which the map remains non-zero upon specialising. As in the proof of Lemma 2.13 in [80], it is possible to define a cocycle $\mathscr{B} : \Gamma \to P_\Sigma[\mathfrak{L}^{-1}] \to \mathbb{F}$. Note that this is not a coboundary as if it were, $\begin{pmatrix} 1 & \mathscr{B} \\ 0 & \overline{\chi} \end{pmatrix}$ would be conjugated to $\begin{pmatrix} 1 & 0 \\ 0 & \overline{\chi} \end{pmatrix}$ and this would contradict the universality of $P_{\Sigma,\Delta}$. The cocycle therefore defines a non-zero class $\mathscr{D}$ in $\mathrm{Ext}^1(\overline{\chi}, 1)$ over $\mathbb{F}$.

Fixing $\mathfrak{L}$, we let $P_{\Sigma,\nabla}$ denote $P_\Sigma[\mathfrak{L}^{-1}]$. Let $R_\Sigma^{\overline{\rho}_\mathscr{D}}$ denote the universal ring for deformations of $\overline{\rho}_\mathscr{D}$ of type $\Sigma$ and $R_\Sigma^{\overline{\rho}_\mathscr{D},\square}$ denote the universal ring for $T$-framed deformations of $\overline{\rho}_\mathscr{D}$ of type $\Sigma$. There is a natural map,

$$P_\Sigma \to R_\Sigma^{\overline{\rho}_\mathscr{D}}$$

given by the pseudo-deformation $\det : R_\Sigma^{\overline{\rho}_\mathscr{D}}[\Gamma] \to R_\Sigma^{\overline{\rho}_\mathscr{D}}$.

Let $R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}}$ denote $R_\Sigma^{\overline{\rho}_\mathscr{D}} \otimes_{P_\Sigma} P_{\Sigma,\nabla}$. By definition, its spectrum is the pull-pack:

$$\begin{array}{ccc} \mathrm{Spec}\, R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}} & \to & \mathrm{Spec}\, P_{\Sigma,\nabla} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R_\Sigma^{\overline{\rho}_\mathscr{D}} & \to & \mathrm{Spec}\, P_\Sigma. \end{array}$$

By Proposition 4.2.2 in [91], any specialisation of $R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}}$ gives rise to an irreducible deformation of $\overline{\rho}_\mathscr{D}$ of $\mathrm{Gal}(F_S/F)$ of type $\Sigma$.

**Lemma 38.** *$P_{\Sigma,\nabla}$ is isomorphic to $R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}}$.*

*Proof.* It suffices to construct a section $R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}} \to P_{\Sigma,\nabla}$ of the natural homomorphism $P_{\Sigma,\nabla} \to R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}}$. It follows from the argument above that $E_\Sigma \otimes_{P_\Sigma} P_{\Sigma,\nabla}$ defines a lifting $\Gamma \to \mathrm{GL}_2(P_{\Sigma,\nabla})$ for $\overline{\rho}_\mathscr{D}$ of type $\Sigma$ over $P_{\Sigma,\nabla}$. Its conjugacy class therefore gives rise to a homomorphism $R_\Sigma^{\overline{\rho}_\mathscr{D}} \to P_{\Sigma,\nabla}$ by the universal property of $R_\Sigma^{\overline{\rho}_\mathscr{D}}$; indeed, this is an $P_\Sigma$-algebra homomorphism. This gives rise, by Stacks Project Lemma 10.9.7 for example, to an $P_\Sigma$-algebra homomorphism $R_{\Sigma,\nabla}^{\overline{\rho}_\mathscr{D}} \to P_{\Sigma,\nabla}$ we seek. □

## 6.4 Reducible non-split $\overline{\rho}$ and cuspidal eigenforms

Let $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation and suppose that its semi-simplification, up to twists, is $\begin{pmatrix} 1 & 0 \\ 0 & \overline{\chi} \end{pmatrix}$ for a totally odd character $\overline{\chi}$. Let $\chi$ denote the Techmuller lifting of $\overline{\chi}$.

Let $S$ denote the union of $S_p$, $S_\infty$ and the set of places in $F$ at which $\chi$ is ramified. Let $U_S$ denote the open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ such that, for every v not in $S$, $U_S \cap \mathrm{GL}_2(F_v) = \mathrm{GL}_2(\mathscr{O}_{F_v})$; for every v in $S - S_p$, $U_S \cap \mathrm{GL}_2(F_v)$ defines the subgroup of matrices in $\mathrm{GL}_2(\mathscr{O}_{F_v})$ that reduce mod the conductor $c_v(\chi)$ at v to the matrices of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$; and for v in $S_p$,

$U_S \cap \mathrm{GL}_2(F_v) = \mathrm{GL}_2(F_v)$. Let $U_S^{[r]}$ denote the subgroup of matrices in $U_S$ such that, for every v in $S_p$, $U_S^{[r]} \cap \mathrm{GL}_2(F_v)$ defines the subgroup of matrices in $\mathrm{GL}_2(\mathscr{O}_{F_v})$ which reduce modulo $\mathfrak{m}_{F_v}^r$ to the unipotent matrices. Let $S_2(U_S^{[r]}, \mathscr{O})$ denote the $\mathscr{O}$-module of cusp forms for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ of parallel weight 2 and of level $U_S^{[r]}$. Let $T_2(U_S^{[r]}, \mathscr{O}) \subset \mathrm{End}(S_2(U_S^{[r]}, \mathscr{O}))$ denote the Hecke algebra generated by $T_v$ for v not lying in $S$ and $U_v = [U_S^{[r]} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} U_S^{[r]}]$ and $S_v = [U_S^{[r]} \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U_S^{[r]}]$ for v in $S_p$.

Letting $e$ denote the Hida idempotent, $eT_2(U_S^{[r]}, \mathscr{O})$ defines an inverse system (with respect to $r$) and we let $eT_2(U_S, \mathscr{O})$ denote the limit.

The diamond operator $\langle\ \rangle : (\mathscr{O}_F/p^r)^\times \to T_2(U_S^{[r]}, \mathscr{O})^\times$, as normalised in [49], extends to $\langle\ \rangle : (\mathscr{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\overline{\mathscr{O}}_F^\times \to T_2(U_S, \mathscr{O})^\times$, while $v \mapsto U_v \in T_2(U_S^{[r]}, \mathscr{O})$ extends to $U : \prod_v F_v^\times \to T_2(U_S, \mathscr{O})$. The 'Hida' nearly ordinary Hecke algebra $eT_2(U_S, \mathscr{O})$ is a finite, torsion-free, algebra over $\Lambda = \Lambda_p \hat{\otimes} \Lambda_p$ via $U|_{\Delta_p} \times \langle\ \rangle|_{\Delta^p}$ (see [49] and [48]). There exists a Galois representation

$$\rho_S : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(eT_2(U_S, \mathscr{O}))$$

which is unramified outside $S$, $\mathrm{tr}\rho_S(\mathrm{Frob}_v) = T_v$ for every v not lying in $S$, $\det \rho_S|_{\Delta^p} = \chi_{\mathrm{cyc}}\langle\ \rangle|_{\Delta^p}$ and, for every place v above $p$,

$$\rho_S|_{\Delta_v} \sim \begin{pmatrix} U|_{\Delta_v} & * \\ 0 & \chi_{\mathrm{cyc}}\langle\ \rangle|_{\Delta_v}(U|_{\Delta_v})^{-1} \end{pmatrix}.$$

**Proposition 39.** *Suppose that the $p$-adic $L$-function $L_p(F, -1, \chi\omega^{-1}) \in \mathscr{O}$ (where $\omega$ denotes the Teichmuller lifting of the mod $p$ cyclotomic character) is divisible by $\lambda$. Then there exists a non-Eisenstein maximal ideal $\mathfrak{m}_S \subset eT_2(U_S, \mathscr{O})$ such that, if $\overline{\rho}_{\mathfrak{m}_S} : \mathrm{Gal}(F_S/F) \to \mathrm{GL}_2(\mathbb{F})$ denote the corresponding Galois representation,*

- $\mathfrak{m}_S$ *contains* $T_v - (1 + \chi(\mathrm{Frob}_v))$ *for evert* v *not lying in* $S$ *and* $U_v - 1$ *for* v *in* $S_p$,
- $\mathfrak{m}_S$ *contains the kernel of* $\chi\omega^{-1} : \Delta^p \to \mathbb{F}$,
- $\overline{\rho}_{\mathfrak{m}_S}$, *hence* $\overline{\rho}$, *is of the form* $\begin{pmatrix} 1 & * \\ 0 & \overline{\chi} \end{pmatrix}$ *with non-zero* $*$.

*Proof.* This is proved in Proposition 3.18 of [80] following Ribet's trick. □

## 6.5 Pro-modularity of irreducible pseudo-deformations over $P_{\Sigma,\nabla}$ when $\overline{\rho}$ is a non-split extension of $\overline{\chi}$ of non-CM type

Fix a class $\mathscr{D}$ in $\mathrm{Ext}^1_{\mathbb{F}[\Delta]}(\overline{\chi}, 1)$ and suppose that it is non-zero. Suppose that $\overline{\rho}_{\mathscr{D}}$ is *not* a (reducible) representation that is induced from an imaginary extension $E$ of $F$ in which every place of $F$ above $p$ splits completely. This is similar to the setting considered in 5.1. The assumption amounts to demanding that $\overline{\rho}_{\mathscr{D}}$ is *not* induced from a character $\overline{\zeta}$ of $\mathrm{Gal}(\overline{F}/E)$ such that the character $\overline{\zeta}_c$, obtained by conjugating $\overline{\zeta}$ by the order 2 generator $c$ of $\mathrm{Gal}(E/F)$, is isomorphic to $\overline{\zeta}$.

Let $\Gamma$ be a co-height one prime of $R^{\overline{\rho}_{\mathscr{D}}}_{\Sigma,\nabla}$ and let $R = R_\Gamma$ denote the normal closure of $R^{\overline{\rho}_{\mathscr{D}}}_{\Sigma,\nabla}/\Gamma$ in the field $K = K_\Gamma$ of fractions. The universal deformation of $\overline{\rho}_{\mathscr{D}}$ of type $\Sigma$ gives rise to

$$\rho : \mathrm{Gal}(F_S/F) \to \mathrm{GL}_2(R)$$

and we assume

- $\rho \otimes_R K$ is irreducible,
- $\det \rho$ is of finite order,
- if $p > 2$, $\rho$ is either non-dihedral or it is dihedral but not $F^+$-dihedral for the quadratic extension $F^+$ of $F$ in $F(\zeta_p)$; while if $p = 2$, $\rho$ is not dihedral,
- $\rho_v$, at v above $p$, is reducible with distinct diagonal characters on the diagonal,
- $\rho_v$ is trivial at every place v in $S_R$,
- $\bar{\rho}_{\mathscr{D}}$ is Eisenstein modular, i.e. there exists $\mathfrak{m}$ such that $\bar{\rho}_{\mathscr{D}} \simeq \bar{\rho}_{\mathfrak{m}}$.

Let $H_{\Sigma_{Q,N}}$ and $T_{\Sigma_{Q,N}}$ denote the Hecke module and Hecke algebra as defined earlier completed at the pre-image $\mathfrak{m}_{Q,N} \subset eT_{\Sigma_{Q,N}}(U_{\Sigma_{Q,N}})$ (an Eisenstein maximal ideal) of $\mathfrak{m}$. Let $H_{\Sigma_{Q,N},\nabla} = H_{\Sigma_{Q,N}} \otimes_{P_{\Sigma_{Q,N}}} P_{\Sigma_{Q,N},\nabla}$ and $T_{\Sigma_{Q,N},\nabla} = T_{\Sigma_{Q,N}} \otimes_{P_{\Sigma_{Q,N}}} P_{\Sigma_{Q,N},\nabla}$.

There is a surjection
$$P_{\Sigma_{Q,N},\nabla} \to T_{\Sigma_{Q,N},\nabla} \subset \mathrm{End}(H_{\Sigma_{Q,N},\nabla})$$
and a natural map
$$P_{\Sigma_{Q,N}} \to R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma_{Q,N}} \to R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma_{Q,N},\nabla}$$
defined by the pseudo-representation associated to the universal deformation of type $\Sigma$ over $R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma_{Q,N}}$.

**Definition**. We say that a prime ideal of $P_{\Sigma,\nabla}$ is pro-modular if it contains $\ker(P_{\Sigma,\nabla} \to T_{\Sigma,\nabla})$. We say that a prime ideal of $R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma,\nabla}$ is pro-modular if its contraction in $R^{\bar{\rho}}_{\Sigma,\nabla}$ is pro-modular.

We have
$$\begin{array}{ccccccc} A^{\square}_{\Sigma} = A^{\square}_{\Sigma_{Q,N}} & \longrightarrow & R^{\bar{\rho}_{\mathscr{D}},\square}_{\Sigma_{Q,N},\nabla} & \longleftarrow & P^{\square}_{\Sigma_{Q,N},\nabla} & \longrightarrow & T^{\square}_{\Sigma_{Q,N},\nabla} \subset \mathrm{End}(H^{\square}_{\Sigma_{Q,N},\nabla}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma,\nabla} & \longleftarrow & P_{\Sigma,\nabla} & \longrightarrow & T_{\Sigma,\nabla} \end{array}$$

The representation $\rho$ define coheight one primes in $\boldsymbol{\mu}_{\Sigma_{Q,N}} \subset R^{\bar{\rho}_{\mathscr{D}},\square}_{\Sigma_{Q,N},\nabla}$ and in $\boldsymbol{\mu} \subset A^{\square}_{\Sigma}$. Since $\Gamma$ is pro-modular, there also is a co-height one prime in $T^{\square}_{\Sigma_{Q,N}}$ that pulls back to $\Gamma$. As in earlier sections, let $R^{\bar{\rho}_{\mathscr{D}},\square,\Gamma}_{\Sigma_{Q,N},\nabla}$, $A^{\square,\Gamma}_{\Sigma}$, $T^{\square,\Gamma}_{\Sigma_{Q,N},\nabla}$, $H^{\square,\Gamma}_{\Sigma_{Q,N},\nabla}$ denote the completed localisations with respect to primes corresponding to $\Gamma$; if $p = 2$, we also have $R^{\bar{\rho}_{\mathscr{D}},\square}_{\Sigma_{Q,N},\nabla}/\nabla_{Q,N}$ and the completed localisation $R^{\bar{\rho}_{\mathscr{D}},\square,\Gamma}_{\Sigma_{Q,N},\nabla}/\nabla_{Q,N}$ with respect to the image of $\boldsymbol{\mu}_{\Sigma_{Q,N}}$ as before. In light of Lemma 38, we similarly have $P^{\square}_{\Sigma_{Q,N},\nabla}/\nabla_{Q,N}$ and $P^{\square,\Gamma}_{\Sigma_{Q,N},\nabla}/\nabla_{Q,N}$.

It follows from Corollary 23 with $R^{\bar{\rho}_{\mathscr{D}}}_{\Sigma,\nabla}$ in place of $R_{\Sigma}$ that we have

$$\begin{array}{ccccccc} & & & & \left(\Lambda\hat{\otimes}_{\mathscr{O}} R^{\square}[[\Delta_Q]]\right)^{\Gamma} & & \\ & & & & \downarrow & & \\ A^{\square,\Gamma}_{\Sigma_Q} = A^{\square,\Gamma}_{\Sigma}[[X_1,\dots,X_r]] & \longrightarrow & R^{\bar{\rho}_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla} & \longleftarrow & P^{\square,\Gamma}_{\Sigma_Q,\nabla} & \longrightarrow & T^{\square,\Gamma}_{\Sigma_Q,\nabla} \subset \mathrm{End}(H^{\square,\Gamma}_{\Sigma_Q,\nabla}) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & R^{\bar{\rho}_{\mathscr{D}},\Gamma}_{\Sigma,\nabla} & \longleftarrow & P^{\Gamma}_{\Sigma,\nabla} & \longrightarrow & T^{\Gamma}_{\Sigma,\nabla} \end{array}$$

if $p > 2$. A similar diagram holds with $R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}/\nabla_Q$ (resp. $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\nabla_Q$, resp. $A^{\square,\nabla,\Gamma}_{\Sigma_Q} = A^{\square,\Gamma}_{\Sigma}[[X_1,\dots,X_{r-(2+q)}]]$) in place of $R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}$ (resp. $P^{\square,\Gamma}_{\Sigma_Q,\nabla}$, resp. $A^{\square,\Gamma}_{\Sigma_Q}$).

**Lemma 40.**

- $H^{\square,\Gamma}_{\Sigma_Q,\nabla}$ *is a free module over* $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}$
- $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\ker(\Lambda\otimes R^{\square}[[\Delta_Q]]\to\Lambda)\simeq H^{\Gamma}_{\Sigma_Q,\nabla}$.
- $A^{\square,\Gamma}_{\Sigma_Q}\to R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}$ *(resp.* $A^{\square,\nabla,\Gamma}_{\Sigma_Q}\to R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}/\nabla_Q$*) is surjective (resp. if* $p=2$*).*

*Proof.* This can be proved exactly as in Lemma 26. ☐

**Proposition 41.** *The map* $P^{\square,\Gamma}_{\Sigma_Q,\nabla}\to R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}$ *is an isomorphism.*

*Proof.* This follows immediately from Lemma 38. ☐

**Theorem 42.** $H^{\square,\Gamma}_{\Sigma_Q,\nabla}$ *is a faithful module over* $P^{\square,\Gamma}_{\Sigma_Q,\nabla}$ *(resp.* $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\nabla_Q$*) and the surjection* $P^{\square,\Gamma}_{\Sigma_Q,\nabla}\to T^{\square,\Gamma}_{\Sigma_Q,\nabla}$ *(resp.* $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\nabla_Q\to T^{\square,\Gamma}_{\Sigma_Q,\nabla}$*) is an isomorphism if* $p>2$ *(resp. if* $p=2$*).*

*Proof.* We assume $p > 2$. Lemma 27 continues to hold in the reducible case, except that the irreducibility of $\rho_\Gamma$, which we assume. Let $\triangle$ be a minimal prime of $\Lambda$.

We firstly observe that, by Lemma 40 and Proposition 41, one may think of Spec $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ as a closed subscheme of Spec $A^{\square,\Gamma}_{\Sigma_Q}$:

$$\operatorname{Spec} P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle\simeq\operatorname{Spec} R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle\hookrightarrow\operatorname{Spec} A^{\square,\Gamma}_{\Sigma_Q}/\triangle.$$

The $P^{\square}_{\Sigma_Q,\nabla}/\triangle$-depth of $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ is greater than and equal to the depth of $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ as a module over $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}/\triangle$. Since $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ is free over $\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}/\triangle$, the latter equals

$$\begin{aligned}
& \dim\left[R^{\square}[[\Delta_Q]]\hat{\otimes}\Lambda\right]^{\Gamma}/\triangle\\
= \; & q+1+[F:\mathbb{Q}]+\gamma_F+4|T|\\
= \; & \dim A^{\Gamma}_{\Sigma_Q}/\triangle\\
\geq \; & \dim R^{\bar\rho_{\mathscr{D}},\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle=\dim P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle.
\end{aligned}$$

As a result, one deduces that the support of $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ is a union of irreducible components of Spec $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$.

When $\zeta$ is distinct, Spec $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is irreducible and it therefore follows that

$$\operatorname{Supp}_{P^{\square}_{\Sigma_Q,\nabla}/\triangle}H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle=P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle=A^{\square,\Gamma}_{\Sigma_Q}/\triangle$$

and that $H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ is also a nearly faithful module over $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ (and over $A^{\Gamma}_{\Sigma_Q}/\triangle$).

From now onwards, suppose that $\zeta$ is trivial. The repeated application of Lemma 2.2 in [83] then proves (the case $\zeta$ is trivial) that

$$\operatorname{Supp}_{P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle}H^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle=P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle=A^{\square,\Gamma}_{\Sigma_Q}/\triangle$$

and that $H^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is a nearly faithful $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$-module. It also follows that $p$ is $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$-regular since $p$ is $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$-regular (Proposition 27 proves that $A^{\square,\Gamma}_{\Sigma_Q}/\triangle$ is Cohen-Macaulay). On the other hand, $A^{\square,\Gamma}_{\Sigma_Q}/\triangle[1/p]$ is reduced and therefore $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle[1/p]$ is reduced, and the $p$-torsion freenes of $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ proves that $P^{\square,\Gamma}_{\Sigma_Q,\nabla}/\triangle$ is reduced.

Applying Lemma 18 to $H^{\square,\Gamma}_{\Sigma_Q,\nabla}$, which is finitely generated with $H^{\square,\Gamma}_{\Sigma_Q,\nabla}[1/p]$ faithful and Cohen-Macaulay over $P^{\square,\Gamma}_{\Sigma_Q,\nabla}$, one concludes that $(A^{\square,\Gamma}_{\Sigma_Q}/J)[1/p] \simeq (P^{\square,\Gamma}_{\Sigma_Q,\nabla}/J)[1/p] \simeq (R^{\bar{\rho}_{\mathcal{D}},\square,\Gamma}_{\Sigma_Q,\nabla}/J)[1/p]$ is reduced. As $P^{\Gamma}_{\Sigma,\nabla} \simeq P^{\square,\Gamma}_{\Sigma_Q,\nabla}/J$ is $p$-torsion free (since $p$ in $P^{\square,\Gamma}_{\Sigma_Q,\nabla}$ remains regular mod $J$, i.e. $p$ is $P^{\Gamma}_{\Sigma,\nabla}$-regular), $P^{\Gamma}_{\Sigma} \simeq R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla}$ are reduced. This concludes a proof of the first set of assertions.

To prove the second assertion, one observes that Lemma 2.2 in [83] proves $H^{\Gamma}_{\Sigma,\nabla} \simeq H^{\square,\Gamma}_{\Sigma_Q,\nabla}/J$ is nearly faithful over $P^{\Gamma}_{\Sigma,\nabla} \simeq P^{\square,\Gamma}_{\Sigma_Q,\nabla}/J$ but, as $P^{\Gamma}_{\Sigma,\nabla}$ is reduced, $H^{\Gamma}_{\Sigma,\nabla}$ is indeed faithful over $P^{\Gamma}_{\Sigma,\nabla}$. The isomorphism $R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla} \simeq T^{\Gamma}_{\Sigma,\nabla}$ holds because its kernel is zero by the faithfulness.

The case when $p = 2$ follows similarly. □

**Proposition 43.** *Suppose that $F$ satisfies the conditions of Proposition 32. Then $P_{\Sigma,\nabla}$ contains an admissible prime $\Gamma$ and*

$$R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla} \simeq P^{\Gamma}_{\Sigma,\nabla} \simeq T^{\Gamma}_{\Sigma,\nabla}$$

*holds. Every prime of $P_{\Sigma,\nabla}$ is pro-modular.*

*Proof.* Let $\mathcal{D}$ be a non-zero co-cycle associated to the universal pseudo-deformation over $P_{\Sigma,\nabla}$. We may argue as in Proposition 32 and Proposition 36 to find a prime $\Gamma$ in $R^{\bar{\rho}_{\mathcal{D}}}_{\Sigma,\nabla} \simeq P_{\Sigma,\nabla}$ satisfying all the conditions for it to be admissible except the irreducibility of the associated representation $\rho_\Gamma : \mathrm{Gal}(F_S/F) \to \mathrm{GL}_2(R)$ where $R$ is isomorphic to $\mathbb{F}[[\pi]]$. However, we may, and will, easily ascertain $\Gamma$ does not contain $\mathfrak{L}$ in $P_{\Sigma,\nabla}$ and the irreducibility of $\rho_\Gamma$ follows from $R$ being a domain. The isomorphism $R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla} \simeq T^{\Gamma}_{\Sigma,\nabla}$ follows as in the Proof of Proposition 25, while the isomorphism $R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla} \simeq P^{\Gamma}_{\Sigma,\nabla}$ is Proposition 41. The second assertion follows exactly as in Corollary 33. □

## 6.6 Pro-modularity of irreducible pseudo-deformations over $P_{\Sigma,\nabla}$ when $\bar{\rho}$ is a non-split extension of $\bar{\chi}$ of CM type

Let $E$ be an imaginary extension of $F$ in which every place of $F$ above $p$ splits completely. Suppose that $\bar{\rho}_{\mathcal{D}}$ is induced from a character $\bar{\zeta}$ of $\mathrm{Gal}(\overline{F}/E)$. Since $\bar{\rho}_{\mathcal{D}}$ is, by assumption, reducible, the conjugate character $\bar{\zeta}_c$ is isomorphic to $\bar{\zeta}$. We may, and will, furthermore assume that the restriction of $\bar{\rho}$ to $\mathrm{Gal}(\overline{F}/E)$ is trivial, and hence assume that $\bar{\chi}$ is trivial upon restriction to $\mathrm{Gal}(\overline{F}/E)$, i.e. $\bar{\chi}$ factors through the character $\mathrm{Gal}(F/E) \to \mathbb{F}^\times$ of order 2 associated to the extension $E$ over $F$. However, since $\bar{\chi}$ is assumed to non-trivial, we may assume $p$ to be odd.

**Proposition 44.** *Suppose that $F$ satisfies the first condition in Proposition 32. Let $\rho^{\mathcal{D}}_{\Sigma,\nabla}$ denote the universal deformation of $\bar{\rho}_{\mathcal{D}}$ of type $\Sigma$ over $R^{\bar{\rho}_{\mathcal{D}}}_{\Sigma,\nabla} \simeq P_{\Sigma,\nabla}$.*

- *If $\rho^{\mathcal{D}}_{\Sigma,\nabla}$ is non-dihedral, then there exists an admissible prime $\Gamma$ of $R^{\bar{\rho}_{\mathcal{D}}}_{\Sigma,\nabla}$ such that $R^{\bar{\rho}_{\mathcal{D}}}_{\Sigma,\nabla} \simeq P_{\Sigma,\nabla} \to T_{\Sigma,\nabla}$ gives rise to an isomorphism*

$$R^{\bar{\rho}_{\mathcal{D}},\Gamma}_{\Sigma,\nabla} \simeq P^{\Gamma}_{\Sigma,\nabla} \simeq T^{\Gamma}_{\Sigma}.$$

- *If $\rho^{\mathcal{D}}_{\Sigma,\nabla}$ is dihedral, then $P_{\Sigma,\nabla}$ is pro-modular.*
- *Every prime of $P_{\Sigma,\nabla}$ is pro-modular.*

*Suppose that $F$ satisfies the conditions of Proposition 32. Every prime of $P_{\Sigma,\nabla}$ is pro-modular.*

*Proof.* This follows as in Proposition 36 with $P_{\Sigma,\nabla} \simeq R^{\bar\rho_{\mathcal{D}}}_{\Sigma,\nabla}$ in place of $R_\Sigma$. □

### 6.7 Reducible split $\bar\rho$ and Eisenstein series, and pro-modularity of reducible deformations over $P_{\Sigma,\Delta}$

For an open compact subgroup $U$ of $G(\mathbb{A}^\infty)$ which is hyper-special maximal compact to $p$, we let $Y_U(\mathbb{C})$ denote the union

$$\bigoplus_J \Gamma(U,J)\backslash(\mathbb{C}-P)^H_+$$

of quotients of $[F:\mathbb{Q}]$-copies of the complex upper half plane $(\mathbb{C}-P)_+$ indexed by $H = \mathrm{Hom}_{\mathbb{Q}}(F,P)$, where $J$ ranges over a fixed set of representatives for the strict ideal class group $\mathbb{A}_F^\times/F^\times(U\cap\mathbb{A}_F^\times)(F\otimes_{\mathbb{Q}} P)^\times_+$ and $\Gamma(U,J)$ denote the corresponding congruence subgroup of $\mathrm{GL}_2(F)_+$. We then let $X_U(\mathbb{C})$ denote its Bailey-Borel-Satake-Serre compactification; the boundary $X_U(\mathbb{C})-Y_U(\mathbb{C})$ is described as the union

$$\bigoplus_J \Gamma(U,J)\backslash\mathbb{P}^1(F).$$

Let $C(\Gamma(U,J))$ denote the set of representatives for $\Gamma(U,J)\backslash\mathbb{P}^1(F)$. Similarly define $C(\Gamma(U^{[r]},J))$ with $U^{[r]}$ in place of $U$. For $c$ in $C(\Gamma(U,J))$, we let $\Gamma_c(U,J)$ denote the stabliser of $c$ in $\Gamma(U,J)$. Similarly define $\Gamma_c(U^{[r]},J)$ for $c$ in $C(\Gamma(U^{[r]},J))$.

We follow Hida [50] (and Harder [44]) to consider the $p$-ordinary 'Eisenstein/boundary cohomology'

$$E^{[r]}_s = \bigoplus_J \bigoplus_{c\in C(\Gamma(U,J))} eH^\bullet(\Gamma_c(U^{[r]},J),\mathcal{O}/\lambda^s)$$

at the degree $\bullet = [F:\mathbb{Q}]$ which corresponds to a system of cusps of level $p^r$ over each $c$ which are 'unramified at (every place of $F$ above) $p$' in the sense of [50]; the degree $\bullet = 0$, on the other hand, corresponds to a system of cusps over $c$ that are (totally) 'ramified at $p$'. The cohomology group comes equipped with natural action of the torus $\begin{pmatrix}(\mathcal{O}_F/p^r)^\times & 0\\ 0 & (\mathcal{O}_F/p^r)^\times\end{pmatrix}$; we choose an isomorphism of the torus with $(\mathcal{O}_F/p^r)^\times\times(\mathcal{O}_F/p^r)^\times$ by sending $\begin{pmatrix} r & 0\\ 0 & s\end{pmatrix}$ to $(r,s)$. Let $E_s$ denote the limit of the direct system $\{E^{[r]}_s\}$ with respect to $r$ (for a fixed $s$) and $E$ the limit of of the direct system $\{E_s\}$ with respect to $s$. Fixing a sufficiently large ideal $N$ of $\mathcal{O}_F$ prime to $p$ such that the principal congruence subgroup of level $N$ is contained in $U$, we let $\Delta^{[r]} = (\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\ker(\mathcal{O}_F^\times\to(\mathcal{O}_F/Np^r)^\times)$. The limit $\Delta^{[\infty]}$ of $\{\Delta^{[r]}\}$ has dimension $1+\gamma_F$ and its torsion-free part is non-canonically isomorphic to the torsion free part of $\Delta^p$. Theorem 3.12 and Theorem 3.14 in [50] establish that the $E$ is computed in terms of:

$$\bigoplus_J \bigoplus_{c\in C(\Gamma(U,\Gamma_J))} C\left(\Delta_c(U,J),L/\mathcal{O}\right)\times C\left(\Delta_c(U,J),L/\mathcal{O}\right),$$

where $C(\Delta_c(U,J), L/\mathcal{O})$ denotes the space of continuous functions (with values in '$L/\mathcal{O}$') defined on the quotient $\Delta_c(U,J)$ of $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z})^\times$ by the closure of the reductive quotient of $\Gamma_c(U,J)$, i.e. the quotient of $\Gamma_c(U,J)$ by the intersection of $\Gamma(U,J)$ and the unipotent radical of the stabiliser of $c$ in $B$. One sees that $E$ is naturally a finitely generated over $\mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right] \hat{\otimes} \mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right]$.

As in [49], [48], [47] and Theorem II of [50], Corollary 3.14in [50] asserts that the subspace of $E$ killed by $(\ker(\chi_{\mathrm{cyc}} \circ \mathrm{Art})^\ell, \ker(\chi_{\mathrm{cyc}} \circ \mathrm{Art})^{k-2+\ell})$ with $(k,\ell) \in \mathbb{Z} \times \mathbb{Z}$ and $k \geq 2$ 'specialises' to the space of Eisenstein series of weight $k$, twisted by $(\chi_{\mathrm{cyc}} \circ \mathrm{Art})^\ell \circ \det$.

**Remark**. Since $U^{[r]} \subset G(\mathbb{A}^\infty)$ is defined to be the subgroup of those congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ mod $p^r$ (rather than those congruent to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ mod $p^r$ as in [50]), the cusps of level $\Gamma^{[r]}$ over $c$ in $C(\Gamma)$ come in pairs, one for unramified and one for (totally) ramified (rather than only seeing the former). Geometrically, this corresponds to 'balanced level structure at $p^r$' in the sense of Katz-Mazur in a neighbourhood of $c$. This viewpoint is consistent with, and expected from, work [51] and [52] of N. Katz in the 70s about constructions of 'Eisenstein measures', for example.

Let $T_2^\Delta(U_S, \mathcal{O})$ denote the subalgebra in the endmorphism ring of $E$ over $\mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right] \hat{\otimes} \mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right]$ (where $U = U_S$ is chosen to be the open compact subgroup of $G(\hat{\mathbb{Z}})$ consisting of those matrices which reduce to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ modulo the prime-to-$p$ conductor of $\chi$) generated over $\mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right] \hat{\otimes} \mathcal{O}\left[\left[\Delta^{[\infty]}\right]\right]$ by $T_v$ for v not lying in $S$ and $U_v$, $S_v$ for v in $S_p$, and let $T_\Delta$ denote the localisation with respect to the maximal ideal of $eT_2^\Delta(U_S, \mathcal{O})$ generated by $T_v - (1 + \overline{\chi}(\mathrm{Frob}_v))$ for v not in $S$ and $U_v - 1$, $S_v - \overline{\chi}(\mathrm{Frob}_v)$ for v in $S_p$. There exists a totally odd continuous lifting

$$\rho_{S,\Delta} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(T_\Delta)$$

of $\overline{\rho}$ which is a direct sum of characters, and is unramified outside $S$, $\mathrm{tr}\rho_{S,\Delta}(\mathrm{Frob}_v) = T_v$ for every v not lying in $S$.

Let $P_\Delta$ denote the reducible quotient of the maximal quotient of the universal ring $P$ for pseudo-deformations of $D_{\mathbb{F}}(\overline{\rho})$ defined by the conditions: if $(D : E \to P, \rho : \Gamma \to E^\times)$ is the universal CH representation of $\Gamma = \mathrm{Gal}(F_S/F)$ over $P$ deforming $D_{\mathbb{F}}(\overline{\rho})$, then $\rho$ satisfies the deformation conditions prescribed by $\Sigma$ at every place of $F$ above $p$. By definition, $P_{\Sigma,\Delta}$ is a quotient of $P_\Delta$.

**Proposition 45.** *The reducible pseudo-deformation quotient $P_{\Sigma,\Delta}$ of $P_\Sigma$ is pro-modular.*

*Proof.* This follows from an isomorphism $P_\Delta \simeq T_\Delta$ which can be established as in Proposition 4.2.5 in [91]. □

**Remark**. If we let $\Lambda_{\Sigma,1}$ (resp. $\Lambda_{\Sigma,2}$) denote the universal $\mathcal{O}$-algebra, of relative dimension $1 + \gamma_F$, for liftings of 1 (resp. $\overline{\chi}$) which are trivial at every place $S_R$ and are trivial (resp. cyclotomic) when restricted to the inertia subgroup at every place in $S_L$, it is possible to establish: $P_{\Sigma,\Delta}$ (in the case $\zeta$ is trivial at every place in $S_R$) is isomorphic to $\Lambda_{\Sigma,1} \hat{\otimes} \Lambda_{\Sigma,2}$ as in the proof of in Proposition 4.2.5 in [91], and is isomorphic to the quotient $T_{\Sigma,\Delta}$ of $T_\Delta$ acting on the $\Lambda$-algebra of a $\Lambda$-adic Eisenstein series, defined similarly, but generated over $\Lambda_{\Sigma,1} \hat{\otimes} \Lambda_{\Sigma,2}$.

**Remark**. The proposition proves that every (globally) split lifting of $\overline{\rho} = 1 \oplus \overline{\chi}$ arises from an Eisenstein series.

# 7 Main theorems

**Theorem 46.** *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathscr{O})$ be a continuous representation of the absolute Galois group of a totally real field $F$ such that*

- *$\rho$ is totally odd,*
- *the image of the inertia subgroup at every finite place of $F$ above $p$ is finite.*
- *$\overline{\rho} = (\rho \bmod \lambda)$ is modular– there exists a cuspidal automorphic representation $\Pi$ of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ whose associated $p$-adic Galois representation is isomorphic to $\overline{\rho}$– when $\overline{\rho}$ is absolutely irreducible; and suppose furthermore than $\overline{\rho}$ is $p$-ordinary modular– $\Pi$ is ordinary at every place of $F$ above $p$– when $p = 2$ and $\overline{\rho}$ is unramified (i.e. trivial) at every infinite place of $F$.*
- *The semi-simplification of $\overline{\rho}$ is not scalar, i.e. not twist-equivalent to the trivial representation.*

*Then there exists a holomorphic modular eigenform of parallel weight $1$ on $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ whose associated $p$-adic representation of $\mathrm{Gal}(\overline{F}/F)$ is isomorphic to $\rho$. In particular, $\rho$ has finite image.*

**Remark**. As mentioned in Introduction, the fourth assumption is unnecessary when $p > 2$. Indeed, in this case, the modularity of $\overline{\rho}$ (the third assumption) forces itself to be ramified/non-trivial at every infinite place of $F$.

*Proof.* We firstly observe that it is possible to replace $F$ by its finite totally real soluble extension of $F$ if necessary to assume $\overline{\rho}$ is of the form described in Section 2.8. In fact, we may choose it so that $\overline{\rho}$ is $p$-ordinary modular, if $\overline{\rho}$ is irreducible with insoluble (resp. soluble) image by virtue of [4], [5], [54], [90] and [87] (resp. Langlands-Tunnell and Hida theory). We secondly observe that there exists a finite totally real soluble extension of $F$ for which the conditions of Proposition 32 hold. To this end, let $\ell$ be a fixed rational prime that does not divide any places in $S$; in particular, $\ell$ is distinct from $p$. We may choose a cyclotomic extension over $F$ of degree a large power of $\ell$ such that the places of $F$ in $S$ all remain inert over any cyclotomic $\ell$-extension of sufficiently large degree; this is possible, because every place of $F$ in $S$ splits completely only at finitely many 'layers' (and otherwise remains inert) over the cyclotomic $\mathbb{Z}_\ell$-extension. We may therefore choose $F'$ in such a way that the conditions of Proposition 32 hold simultaneously. We shall call the resulting cyclotomic extension $F$ again.

When $\overline{\rho}$ is reducible, we follow the argument in the proof of Theorem A in [80], with Lemma 11 in [23] and a construction (by [26], say) of $p$-adic $L$-function over totally real fields as our input, one finds an abelian totally real extension $F'$ of $F$ such that $L_p(F', -1, \chi\omega^{-1}) \in \mathscr{O}$ is divisible by $\lambda$– this is the assumption in Proposition 39 to show that there exists a non-Eisenstein maximal ideal of a Hecke algebra corresponding to $\overline{\rho}$.

Granted, we may assume:

- $\overline{\rho}$ satisfies the assumptions in Section 2.8 while maintaining the third and fourth assumptions on $\overline{\rho}$ in the statement of the theorem; in particular, there exists a finite set of places $S = S_p \cup S_R \cup S_L \cup S_A \cup S_\infty$ such that $\overline{\rho}$ is unramified outside $S$ and is trivial at $S_p \cup S_R \cup S_L$ and the image of complex conjugation by $\overline{\rho}$ at every place in $S_\infty$ has determinant $-1$ (whether $p > 2$ or not);

- $\rho$ is totally odd and is unramified outside $S$;
- $\rho$ is unramified at every place in $S_p$ and there exists a partition $S_{p,\mathrm{d}}$ and $S_{p,\mathrm{e}}$ of $S_p$ such that $\rho(\mathrm{Frob}_\mathrm{v})$ has distinct (resp. equal) eigenvalues $\{\alpha_\mathrm{v}, \beta_\mathrm{v}\}$ (resp. $\alpha_\mathrm{v} = 1$) if v lies in $S_{p,\mathrm{d}}$ (resp. $S_{p,\mathrm{e}}$).

Given a subset $\Delta = \Delta_\mathrm{d} \cup \Delta_\mathrm{e} \subseteq S_{p,\mathrm{d}} \cup S_{p,\mathrm{e}} = S_p$, the representation $\rho$ gives rise to a map $\rho_\Delta : R_\Sigma \to \mathscr{O}[\epsilon]$, where $\mathscr{O}[\epsilon]$ is the completed tensor product of the ring of dual numbers $\mathscr{O}[\epsilon_\mathrm{v}]$ as v ranges over $S_{p,\mathrm{e}}$, when we choose a root of the characteristic polynomial of $\rho(\mathrm{Frob}_\mathrm{v})$ to be $(1+\epsilon_\mathrm{v})\alpha_\mathrm{v}$ (resp. $\alpha_\mathrm{v}$, resp. $\beta_\mathrm{v}$) if v lies in $S_{p,\mathrm{e}}$ (resp.$S_{p,\mathrm{d}} - \Delta_\mathrm{d}$, resp. $\Delta_\mathrm{d}$).

Suppose that $\rho$ is irreducible. It then follows from Propositions 25, 32, 36, 43 and 44 that there exists a map

$$F_\Delta : T_\Sigma \to \mathscr{O}[\epsilon]$$

for every $\Delta \subset S_p$ such that

- $T_\mathrm{v}\, F_\Delta = \mathrm{tr}\, \rho(\mathrm{Frob}_\mathrm{v}) F_\Delta$ for every v not $S$;
- $U_\mathrm{v}\, F_\Delta = \alpha_\mathrm{v} F_\Delta$ if v lies in $S_{p,\mathrm{d}} - \Delta_\mathrm{d}$, while $U_\mathrm{v} F_\Delta = \beta_\mathrm{v} F_\Delta$ if v lies in $\Delta_\mathrm{d}$;
- $U_\mathrm{v}\, F_\Delta = \alpha_\mathrm{v} F_\Delta + F_{\Delta - \{\mathrm{v}\}}$ if v lies in $\Delta_\mathrm{e}$ while $U_\mathrm{v} F_\Delta = \alpha_\mathrm{v} F_\Delta$ if v lies in $S_{p,\mathrm{e}} - \Delta_\mathrm{e}$.

as in Theorem 54 in [75]. For example, when $\rho_\Delta$ is irreducible with non-dihedral image (resp. with dihedral image), one can find an admissible prime $\Gamma$ of $R_\Sigma$ which contains the prime ideal corresponding to $\rho_\Delta$, and $F_\Delta$ is given by $T_\Sigma \to T_\Sigma^\Gamma \simeq R_\Sigma^\Gamma \to \mathscr{O}[\epsilon]$ (resp. $T_\Sigma \to R_\Sigma \to \mathscr{O}[\epsilon]$).

Exactly as in the proof of Theorem 54 in [75], we then establish that the $F_\Delta$ define cuspidal overconvergent modular eigenforms of weight one and of 'level Iwahori at $p$', after possibly increasing their levels at $S$ but away from $p$ to assume the eigenvalues at these places are all zero. The argument in Section 6.5 in [75] proves that they 'glue' to define a classical weight one form of level old at $p$ whose associated Galois representation is $\rho$. This proves the modularity of $\rho$.

If $\rho$ is reducible, then $\rho$ arises from an Eisenstein series of parallel weight 1 (c.f. Proposition 45). □

By making appeal to pro-modularity results and Hida theory, we also obtain the following theorem:

**Theorem 47.** *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathscr{O})$ be a continuous representation of the absolute Galois group of a totally real field $F$ such that*

- *$\rho$ is totally odd,*
- *the restriction of $\rho$ to the decomposition subgroup at every finite place* v *of $F$ above $p$ is reducible, and is potentially semi-stable with (distinct) Hodge-Tate weight $(k_\tau + \ell_\tau - 1, \ell_\tau)$ at $\tau$ in $H_\mathrm{v} = \mathrm{Hom}_{\mathbb{Q}_p}(F_\mathrm{v}, L)$ for a pair of integers $k_\tau \geq 2$ and $\ell_\tau \geq 0$,*
- *$\overline{\rho} = (\rho \bmod \lambda)$ is modular– there exists a cuspidal automorphic representation $\Pi$ of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ whose associated $p$-adic Galois representation is isomorphic to $\overline{\rho}$– when $\overline{\rho}$ is absolutely irreducible; and suppose furthermore than $\overline{\rho}$ is $p$-ordinary modular– $\Pi$ is ordinary at every place of $F$ above $p$– when $p = 2$ and $\overline{\rho}$ is unramified (i.e. trivial) at every infinite place of $F$.*

- *The semi-simplification of $\overline{\rho}$ is not scalar, i.e. not twist-equivalent to the trivial representation.*

*Then there exists a holomorphic $p$-ordinary modular eigenform of weight $(k, \ell)$ on $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, where $k = \sum_{\tau \in H} k_\tau \tau$ and $\ell = \sum_{\tau \in H} \ell_\tau \tau$ and $H = \coprod_v H_v = \mathrm{Hom}_{\mathbb{Q}_p}(F, L)$, whose associated $p$-adic representation of $\mathrm{Gal}(\overline{F}/F)$ is isomorphic to $\rho$.*

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