The DS conjectures

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Motivation

Let p be a rational prime. Given an odd continuous representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

Serre (as documented by Edixhoven) conjectured: there should be

- $k = k(\overline{\rho}) \geq 1$,
- $N = N(\overline{\rho}) \geq 1$ the prime-to-p Artin conductor,
- an eigenform f in $H^0(\overline{X}_{\Gamma_1(N)}, \omega^k(-Z_{\Gamma_1(N)}))$,

$$Z_{\Gamma_1(N)} = X_{\Gamma_1(N)} - Y_{\Gamma_1(N)},$$

whose associated mod p Galois representation $\overline{\rho}_f$ is isomorphic to $\overline{\rho}$.

$k(\overline{\rho})$

The predicted weight $k(\overline{\rho})$ depends only on $\overline{\rho}|_{D_p}$ and is defined purely in terms of $\overline{\rho}$ (as in Hanneke's talk).

If we assume $\overline{\rho}$ is modular, we can see a few glimpses of the geometric picture I would like to promote today.

$k(\overline{\rho})$ is minimal

Theorem (Coleman-Voloch/Gross/Edixhoven/Serre)

Suppose that there exists a cuspidal eigenform f as above such that $\overline{\rho}_f \simeq \overline{\rho}$. Then

$$k(\overline{\rho}) = w(f)$$

where $w(f)$ is the Hasse weight filtration:

$$w(f) = \min\{k - (p-1)v \mid H^{-v} f \in H^0(\overline{X}_{\Gamma_1(N_f)}, \omega^{k-(p-1)v})\}$$

(as in Payman's talk) and $k(\overline{\rho})$, as a result, is the minimal possible weight of mod p cuspidal eigenforms that give rise to $\overline{\rho}$.

Remark

The proof makes appeal to the idea of θ-cycles:

$$\{w(\theta(f)), \dots, w(\theta^{p-1}(f))\}$$

subject to

- $w(\theta^p f) = w(\theta f)$,
- $w(\theta(f)) \leq w(f) + p + 1$ and the equality holds if and only if p does not divide $w(f)$.

Today

The DS paper is in some sense motivated to generalise the theorem to a (general) totally real field. Issues to overcome that I will highlight today are as follows:

- Lifting mod p HMFs to char 0 is no longer just about weight 1 forms.
- Mod p modular Galois representations.
- Partial weight one forms.
- Compatibility with other mod p modularity of $\overline{\rho}$ (i.e., BDJ/algebraic modularity of $\overline{\rho}$).

A set of notation

Fix $\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_p, \overline{\mathbb{F}}_p$,

$$\imath : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p,$$

$$\jmath : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Let F be a totally real field. Assume $[F : \mathbb{Q}] = d > 1$ and p is inert in F for simplicity (DS-II deals with the general ramified case). Let $\mathbb{F}$ denote the residue field $\mathscr{O}_F/p$.

Let

$$\Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}});$$

$$\imath \circ \Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p) \simeq \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}, \overline{\mathbb{F}}_p)$$

$$\jmath \circ \Sigma = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$$

The Frobenius Φ acts on $\imath \circ \Sigma$. If we fix τ in $\imath \circ \Sigma$,

$$\imath \circ \Sigma = \{\tau, \Phi \circ \tau, \ldots, \Phi^{f-1} \circ \tau\} \simeq \mathbb{Z}/f\mathbb{Z}$$

where $|\mathbb{F}| = p^f$.

Finally, let

$$G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2.$$

By slight abuse of notation, it will also mean its model:

$$G(\mathbb{F}_p) = \mathrm{GL}_2(\mathbb{F}).$$

A quick recap

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be totally odd and continuous.

Definition

Let V be a finite-dimensional $\overline{\mathbb{F}}_p$-representation of $G(\mathbb{F}_p)$. We say that $\overline{\rho}$ is algebraic modular of weight V if there exist

- a quaternion algebra D over F split at p and ramified at all but one place in $\jmath \circ \Sigma$,
- a sufficiently small open compact subgroup $U \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ ($\rightsquigarrow$ the Shimura curve Y_U over F) such that
  - U is of level prime to p (i.e. $G(\mathbb{Z}_p) \subset U$),
  - if $U^+ = \ker(U \to G(\mathbb{F}_p))$, then $Y_{U^+} \to Y_U$ is étale of degree equal to $|G(\mathbb{F}_p)|$

such that $\overline{\rho}$ is an $\overline{\mathbb{F}}_p[\mathrm{Gal}(\overline{F}/F)]$-subquotient of $H^1_{\text{ét}}(Y_U \times \overline{F}, \mathcal{V})(1)$.

Lemma

$\overline{\rho}$ is algebraic modular of weight V (as above) if and only if $\overline{\rho}$ is algebraic modular of weight some Jordan-Holder factor of V.

A finite-dimensional irreducible $\overline{\mathbb{F}}_p$-representation V of $G(\mathbb{F}_p)$ is of the form

$$V_{k,\ell} = \bigotimes_{\tau \in \imath \circ \Sigma} \det^{\ell_\tau} \mathrm{Sym}^{k_\tau - 2}(V_{st} \otimes_{\mathbb{F},\tau} \overline{\mathbb{F}}_p),$$

for $(k, \ell) \in \mathbb{Z}^{\imath\circ\Sigma} \times \mathbb{Z}^{\imath\circ\Sigma}$, where $0 \le \ell_\tau, k_\tau - 2 \le p - 1$ but the ℓ_τ are not simultaneously $p - 1$ for all τ, and is often called a Serre weight.

The Buzzard-Diamond-Jarvis conjecture

Given $\overline{\rho}$, the BDJ paper defines a set of Serre weights

$$W(\overline{\rho})$$

as in Fred's talk (an example in the case $[F : \mathbb{Q}] = 2$) and conjectures:

Conjecture (BDJ)

$W(\overline{\rho}) = \{\text{Serre weight } V \mid \overline{\rho} \text{ is algebraic modular of weight } V\}$.

Work of Gee et.al.

Theorem (Gee-Liu-Savitt,....)

Suppose that $p > 2$. The BDJ conjecture holds if

- $\overline{\rho}$ is modular,
- $\overline{\rho}$ is irreducible when restricted to $\mathrm{Gal}(\overline{F}/F(\zeta_p))$,
- when $p = 5$, the projective image of $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is not A_5.

When p is unramified in F, GLS (JAMS) proves the BDJ conjecture in the unitary case. In this case, the BDJ conjecture is deduced by Gee-Kisin/Newton via 'mod p Langlands transfer (from unitary to quaternionic)'.

Set-up for the DS conjecture

Let Γ be an open compact subgroup of $G(\mathbb{A}^\infty)$ that is maximal compact hyperspecial at p and 'sufficiently small'. There exist

- an integral model Y_Γ over $\overline{\mathbb{Z}}_p$ of level Γ for

$$Y_\Gamma(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/\Gamma(\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}))^\Sigma$$

- an automorphic bundle $\mathscr{A}_{(k,\ell)}$ associated to $(k,\ell) \in \mathbb{Z}^{\imath\circ\Sigma} \times \mathbb{Z}^{\imath\circ\Sigma}$:

$$\mathscr{A}_{(k,\ell)} = \bigotimes_\tau \omega_\tau^{k_\tau} \otimes \delta_\tau^{\ell_\tau}$$

where

$$\omega = s_*\Omega_{A/Y_\Gamma} = \bigoplus_\tau \omega_\tau$$

and

$$\delta = \bigwedge^2_{\mathscr{O}_F \otimes \mathbb{Z}_p \mathscr{O}_{Y_\Gamma}} R^1 s_* \Omega^\bullet_{A/Y_\Gamma} = \bigoplus_\tau \delta_\tau .$$

Example ($\ell = 0$)

In stark contract to the case $F = \mathbb{Q}$, there are lots of mod p Hilbert modular forms of 'negative weights'.

Example

For every τ in $\imath \circ \Sigma$, let $H_\tau \in H^0(Y_\Gamma \times \overline{\mathbb{F}}_p, \omega^p_{\Phi^{-1}\circ\tau} \otimes \omega_\tau^{-1})$ denote the partial Hasse invariant at τ of weight

$$h_\tau = (0, \dots, 0, p, -1, 0, \dots, 0)$$

where p (resp. -1) sits at $\Phi^{-1} \circ \tau$ (resp. τ).

Mod p modular Galois representations

Theorem (DS)

Let f be an element $H^0(Y_\Gamma \times \overline{\mathbb{F}}_p, \mathscr{A}_{(k,\ell)})$ and S be a finite set of finite places in F, containing all v dividing p and all v such that $\mathrm{GL}_2(\mathscr{O}_{F_v}) \not\subset \Gamma$.
Suppose that

$$T_v f = \alpha_v f$$

and

$$S_v f = \beta_v f$$

for all v not in S. Then there exists a continuous representation

$$\overline{\rho}_f : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is unramified outside S and the characteristic polynomial in X of $\overline{\rho}_f(\mathrm{Frob}_v)$ is

$$X^2 - \alpha_v X + \beta_v \mathbf{N}_{F/\mathbb{Q}}(v).$$

Remark

The novelty of our theorem is that (k, ℓ) does not have to satisfy the parity condition that $k_\tau + 2\ell_\tau$ is independent of τ in Σ. The parity case is known by Emerton-Reduzzi-Xiao and Goldring-Koskivirta.

Idea of our proof

Recall from Fred's remark: not every $\overline{\rho}$ arises as the reduction of a characteristic zero eigenform of level prime to p (unless it is expected to be modular of paritious weight).

The idea then is to establish congruences, i.e., find an eigenform of parallel weight $N+2$ (to be specified) and level $\Gamma \cap \Gamma_1(p)$ which is congruent mod p to f and which can be lifted to a characteristic zero eigenform when N is sufficiently large (the ampleness of a line bundle over $X_{\Gamma\cap\Gamma_1(p)}$).

The '$+2$' in weight comes from a dualising/regular differential sheaf and '$+N$' is a piece of apparatus needed for lifting forms of weight that is not regular and not paritious.

Preliminary reduction steps

• Γ may be taken to be a principal congruence subgroup of level prime to p,

• by twisting if necessary, WLOG $\ell_\tau = -1$ for every τ in $\imath \circ \Sigma$,

• (can always) find $N \in \mathbb{Z}$ and $r \in \mathbb{Z}^{\imath\circ\Sigma}$ such that $0 \leq r_\tau \leq p-1$ but not all r_τ are simultaneously $p-1$ such that $k-(N+2-r)$ is a linear combination of h_τ's. Multiplying H_τ's defines an injection of global sections $\rightsquigarrow k = N+2-r$.

[• the Katz-Mazur-Pappas covering

$$\pi : Y_{\Gamma\cap\Gamma_1(p)} \to Y_\Gamma$$

with extension

$$\pi : X_{\Gamma\cap\Gamma_1(p)} \to X_\Gamma$$

to minimal compactifications]

- observe that there is a Hecke equivariant injection

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,-1)}) \hookrightarrow H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})$$

with its image contained in

$$H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota_* K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \quad \subset \quad H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota}_* \overline{K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) = H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})$$

where

- $\iota : Y_{\Gamma\cap\Gamma_1(p)} \hookrightarrow X_{\Gamma\cap\Gamma_1(p)}$,
- K denotes the dualising sheaf over the Cohen-Macaulay scheme $Y_{\Gamma\cap\Gamma_1(p)}$,
- and $\omega = \mathscr{A}_{(1,0)}$ is the ample line bundle $\pi^*(\bigotimes_\tau \omega_{X_{\Gamma,\tau}})$.

Interlude: where does this injection come from? There is a component

$$\begin{array}{ccc} \overline{Y}^R_{\Gamma\cap\Gamma_1(p)} & \overset{\gamma}{\hookrightarrow} & \overline{Y}_{\Gamma\cap\Gamma_1(p)} \\ \downarrow & & \downarrow \\ \overline{Y}^R_{\Gamma\cap\Gamma_0(p)} & \overset{(A,\ker F)\leftarrow A}{\simeq} & \overline{Y}_{\Gamma} \end{array}$$

which may be described in terms of Raynaud bundles:

$$\overline{Y}^R_{\Gamma\cap\Gamma_1(p)} = \underline{\mathrm{Spec}}\left(\mathrm{Sym}_{\mathscr{O}_{\overline{Y}_\Gamma}}\left(\bigoplus_{\tau\in\jmath\circ\Sigma}\overline{\omega}_\tau\right) / \left\langle \{\overline{\omega}_\tau^p\}_{\tau\in\jmath\circ\Sigma}, \otimes_\tau\overline{\omega}_\tau^{p-1}\right\rangle\right).$$

The push-forward is a union of automorphic bundles over $\overline{Y}_\Gamma$:

$$(\pi\circ\gamma)_*\mathscr{O}_{\overline{Y}^R_{\Gamma\cap\Gamma_1(p)}} \simeq \bigoplus_r \mathscr{A}_{(r,0)}$$

where r ranges over $\mathbb{Z}^{\imath\circ\Sigma}$ such that $0\le r_\tau\le p-1$ but the r_τ are not simultaneously $p-1$.

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Theorem (DKS)

$$R^r \pi_* K = 0$$

for $r > 0$.

• the theorem implies $R^1\pi_*(\iota_* K) = 0$, hence $\pi_*(\iota_* K) \to \pi_*(\overline{\iota_* K})$ is surjective and, combined with the ampleness of ω, it follows that

$$\begin{array}{ccc} H^0(X_\Gamma, \pi_*\iota_* K \otimes_{\mathscr{O}_{X_\Gamma}} \omega^N) & \longrightarrow & H^0(X_\Gamma, \pi_*(\overline{\iota_* K}) \otimes_{\mathscr{O}_{X_\Gamma}} \overline{\omega}^N) \\ \| & & \| \\ H^0(X_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)}) & \longrightarrow & H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota_* K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \end{array}$$

is surjective when N is sufficiently large, because

$$H^1(X_\Gamma, \pi_*\iota_* K \otimes_{\mathscr{O}_{X_\Gamma}} \omega^N) = 0.$$

• there exists a Galois representation associated to an eigenform in $H^0(X_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})[1/p]$ by work of Carayol, Taylor,....

The folklore conjecture

Conjecture (DS)

Every irreducible, continuous, totally odd representation

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is modular (of some weight) in our sense.

How can we make this statement, i.e., the weight part, more precise? We will do this in terms of the behaviour of $\overline{\rho}_p = \overline{\rho}|_{D_p}$.

In preparation for formulating the DS conjecture (the DK cone)

Definition (Diamond-Kassaei cones)

Let

$$\Xi = \{k = \sum_{\tau \in \imath \circ \Sigma} k_\tau \tau \mid pk_\tau \geq k_{\Phi^{-1} \circ \tau}\} \subset \mathbb{Z}^{\imath \circ \Sigma}$$

and

$$\Xi^+ = \{k \in \Xi \mid k_\tau \geq 1\}.$$

Define

$$k \succeq k'$$

if $k - k'$ is a non-negative linear combination of the weights h_τ of the partial Hasse invariants.

Conjecture

Conjecture (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\imath \circ \Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ^+ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \succeq k(\overline{\rho}, \ell)$
- if $k \in \Xi^+$, then $k \succeq k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_p)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

Remark

Assuming $\overline{\rho}$ is (geometrically) modular, the existence of $k(\overline{\rho}, \ell)$ is suggested by DK– the weight filtration $w(f)$ of a mod p HMF f lies in Ξ (as explined in Payman's talk).

The first condition: 'minimality'

Assuming the existence of $k(\overline{\rho}, \ell) \in \Xi^+$ satisfying the first condition, one sees that

- the conjecture implies the folklore conjecture earlier,
- the conjecture (i.e. the second condition) boils down to

Conjecture (DS)

If $k \in \Xi^+$, then $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $\overline{\rho}_p$ has crystalline lift of weight (k, ℓ).

The second condition: p-adic Hodge-theory

The second condition is suggested by the Breuil-Mézard conjecture and modular representation theory of $G(\mathbb{F}_p)$– it is the underlying theme of DKS.

The qualification '$k \in \Xi^+$' in the second condition is needed– if $k \notin \Xi^+$, the condition $k \succeq k(\overline{\rho}, \ell)$ does not imply that $\overline{\rho}_p$ has crystalline lift of weight (k, ℓ).

Algebraic modularity = geometric modularity

Conjecture (DS)

Let $(k, \ell) \in \mathbb{Z}^{\imath\circ\Sigma} \times \mathbb{Z}^{\imath\circ\Sigma}$ and $k_\tau \geq 2$ for every τ in $\imath \circ \Sigma$. If $\overline{\rho}$ is algebraic modular of weight (k, ℓ), i.e., of weight

$$V_{k,1-k-\ell} = \bigotimes_{\tau} \det{}^{1-k_\tau-\ell_\tau} \mathrm{Sym}^{k_\tau-2}(V_{st} \otimes_\tau \overline{\mathbb{F}}_p),$$

then $\overline{\rho}$ is modular of weight (k, ℓ).

Furthermore, if $k \in \Xi^+$, the converse holds.

Remarks

We know that if $\overline{\rho}$ is algebraic modular of paritious weight (k, ℓ), then $\overline{\rho}$ is modular of weight (k, ℓ).

By our construction of modular Galois representations, if $\overline{\rho}$ is modular of some weight, $\overline{\rho}$ is algebraic modular of some weight.

The DKS paper was motivated by this conjecture.

Example 1: $F = \mathbb{Q}$ and $\ell = 0$

$$\Xi^+ = \{k \geq 1\},$$
$$k \succeq k' \text{ if } k - k' = (p-1)v$$

for a non-negative integer v.

There exists $k(\overline{\rho}) \geq 1$ such that the following are equivalent: for every $k \geq 1$,

- $\overline{\rho}$ is modular of weight k,
- $k \succeq k(\overline{\rho})$,
- $\overline{\rho}_p$ has a crystalline lift of weight $(k-1, 0)$,

$\rightsquigarrow$ $k(\overline{\rho})$ is the smallest possible weight for which $\overline{\rho}$ is modular (see Hanneke's talk).

Example 2: $[F:\mathbb{Q}] = 2$ and $\ell = 0$

Fix τ in $\imath \circ \Sigma$. Let

$$\imath \circ \Sigma = \{\tau, \Phi \circ \tau = \Phi^{-1} \circ \tau\}.$$

Theorem (DS)

Let $2 \leq r \leq p$ and suppose that r is odd. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular.

If $\overline{\rho}_p$ has crystalline lift of weight

$$((r,1),(0,0))$$

(=HT weight $((r-1,0),(0,0))$) then $\overline{\rho}$ is modular of weight

$$((r,1),(0,0)).$$

[τ-labelled weights are always 'on the left']

Proof (Sketch)

For brevity, we only sketch out proof when $2 < r$; the case $r = 2$ is similar. We furthermore assume that $\overline{\rho}$ is of Taylor-Wiles type–the exceptional case can be dealt with by an ad hoc argument.

Step 1

Given that $\overline{\rho}_p$ has crystalline lift of weight

$$((r,1),(0,0)),$$

we deduce from p-adic Hodge theory that $\overline{\rho}_p$ also has crystalline lifts of weight

$$(k,\ell) = ((r-1,p+1),(0,0))$$

and

$$(k',\ell') = ((r+1,p+1),(-1,0)).$$

'p-adic Hodge theory'?

For example, suppose $\overline{\rho}_p$ is reducible and is of the form $\begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$ where

- $\overline{\chi}_1|_I = 1$ (i.e. unramified),
- $\overline{\chi}_2|_I = \lambda_\tau^{1-r}$, where λ_τ denotes the τ-fundamental character,
- $\overline{\chi} = \overline{\chi}_1/\overline{\chi}_2$ has a crystalline lift $\chi = \chi_{(1-r,0)}$ of Hodge-Tate type $(1-r, 0)$ and $\overline{\mathrm{BK}}_\chi$ contains $*$, where

$$\begin{array}{ccccc} H^1(F_p, \overline{\mathbb{Q}}_p(\chi)) & \leftarrow & H^1(F_p, \overline{\mathbb{Z}}_p(\chi)) & \rightarrow & H^1(F_p, \overline{\mathbb{F}}_p(\chi)) \\ \cup & & \cup & & \cup \\ H^1_{\mathrm{BK}}(F_p, \overline{\mathbb{Q}}_p(\chi)) & \leftarrow & \mathrm{BK}_\chi & \rightarrow & \overline{\mathrm{BK}}_\chi \end{array}$$

Then it follows from

$$\lambda_\tau^{r-1} = \lambda_\tau^{r-2} \lambda_{\Phi\circ\tau}^{p}$$

and

$$\lambda_\tau^{r-1} = \lambda_\tau^{r} \lambda_{\Phi\circ\tau}^{-p}$$

that $\overline{\chi}$ has crystalline lifts $\chi_{(2-r,-p)}$ and $\chi_{(-r,p)}$ of Hodge-Tate type

$$(2-r,-p)$$

and

$$(-r,p)$$

respectively, and one can check in both cases that

$$* \in \overline{\mathrm{BK}}_{\chi_{(2-r,-p)}} = H^1(F_p, \overline{\mathbb{F}}_p(\chi_{(2-r,-p)}))$$

and

$$* \in \overline{\mathrm{BK}}_{\chi_{(-r,p)}} = \overline{\mathrm{BK}}_{\chi}.$$

Step 2: algebraic companion forms

By work of Gee and his collaborators on the BDJ conjecture that $\overline{\rho}$ is algebraic modular of weight (k, ℓ) (=Serre weight $V_{k,1-k-\ell}$) and (k', ℓ').

Step 3: geometric companion forms

Since r and 1 are paritious, $\overline{\rho}$ is geometric modular of weight (k, ℓ) and (k', ℓ').

Step 4: combinatorics

Let f (resp. f') be a geometric HMF of weight (k, ℓ) (resp. (k', ℓ')) such that $\overline{\rho}_f \simeq \overline{\rho}$ (resp. $\overline{\rho} \simeq \overline{\rho}_{f'}$). One observes

- $\theta_\tau(f)$ is of weight

  $$((r-1, p+1), (0,0)) + ((1,p), (-1,0)) = ((r, 2p+1), (-1,0)),$$

- $f' H_\tau$ is an eigenform of weight

  $$((r+1, p+1), (-1,0)) + ((-1,p), (0,0)) = ((r, 2p+1), (-1,0)),$$

- $\theta_\tau(f) = f' H_\tau$.

θ-operators/cycles

Theorem (AG)

For any HMF f of weight (k, ℓ), the image $\theta_\tau(f)$ is divisible by H_τ if and only if f is divisible by H_τ or p divides k_τ.

Step 5: a mod p HMF of partial weight one

Deduce from the theorem that f is divisible by H_τ. The HMF f/H_τ of weight

$$((r-1,p+1),(0,0))-((-1,p),(0,0))=((r,1),(0,0))$$

is what we are looking for.