A Serre weight conjecture for mod p Hilbert modular forms

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Before I begin

Everything I say today is joint work with F. Diamond (DS). Some of it is joint with P. Kassaei (DKS).

My notes below contain a lot more details than what I intend to say.

Introduction

Let p be a rational prime. Let

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, odd and irreducible representation over $\overline{\mathbb{F}}_p$.

Serre's conjecture

J.-P. Serre (1987) defined/specified

- $k(\overline{\rho}) \geq 2$
- $N(\overline{\rho}) \geq 1$, the Artin conductor prime to p,

and conjectured that there should be a cuspidal modular eigenform f of weight $k(\overline{\rho})$ and level $N(\overline{\rho})$ such that (for a choice of $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$),

$$f \rightsquigarrow \overline{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_f} \mathrm{GL}_2(\overline{\mathbb{Z}}_p) \twoheadrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is isomorphic to $\overline{\rho}$.

Serre's conjecture in the Hilbert case

Generalising the weight part of Serre's conjecture ("if $\overline{\rho}$ is modular, of what weight exactly?") to the setting of mod p HMFs (of regular weights) was initiated by Buzzard-Diamond-Jarvis (2010).

The BDJ conjecture ($p > 2$) has been proved almost completely by Gee, Liu and Savitt.

Today, I will formulate a Serre weight conjecture for all weights within geometric theory of Hilbert modular forms (Katz, Goren, Andreatta-Goren,...).

We are generalising Edixhoven's reformulation (1992) of Serre's conjecture.

Geometry (of Shimura varieties) seems to provide genuinely new input into the mix.

To start with

Fix $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{F}}_p$ and fix an embedding

$$\imath : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$$

once for all.

Let

$$\Sigma = \imath \circ \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p).$$

Suppose that there is only one prime v in $\mathscr{O}_F$ above p.

Let $\mathbb{F}_v = \mathscr{O}_F / v$ denote the residue field at v.

If $F_v^r \subset F_v$ denote the maximal unramified extension of $\mathbb{Q}_p$ of degree f_v, then the restriction to F_v defines a surjection

$$\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p) = \Sigma \twoheadrightarrow \Sigma^r = \mathrm{Hom}_{\mathbb{Q}_p}(F_v^r, \overline{\mathbb{Q}}_p) \circlearrowleft \phi.$$

Suppose that $\{\tau_\beta(1), \ldots, \tau_\beta(e_v)\} \subset \Sigma$ is the pre-image of β in Σ^r.

Define an index shift Φ on Σ:

$$\cdots \rightsquigarrow \tau_{\phi^{-1}\circ\beta}(e_v) \overset{\phi}{\rightsquigarrow} \tau_\beta(1) \rightsquigarrow \cdots \rightsquigarrow \tau_\beta(e_v) \overset{\phi}{\rightsquigarrow} \tau_{\phi\circ\beta}(1) \rightsquigarrow \cdots$$

Models of HMFs

Let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$$

and let

- $\Gamma \subset G(\mathbb{A}^\infty)$ maximal compact hyperspecial at p, which we always assume sufficiently small,
- (Rapoport/Deligne-Pappas/Pappas-Rapoport) a smooth integral model Y_Γ over $\overline{\mathbb{Z}}_p$ for

$$G(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R})^\Sigma \times G(\mathbb{A}^\infty)/\Gamma$$

Pappas-Rapoport

The Pappas-Rapoport integral model Y_Γ parametrises HBAVs $s : A \to S$ of PEL type (in the sense of DP) such that the locally free $\mathscr{O}_S$-module of rank $[F : \mathbb{Q}] = d = e_v f_v$ with $\mathscr{O}_F$ action

$$\omega = s_* \Omega_{A/S} = \bigoplus_{\beta \in \Sigma^r} \omega_\beta$$

where ω_β (locally free $\mathscr{O}_S$-module of rank e_v) comes equipped with a filtration (a 'complete flag')

$$0 = \omega_\beta(0) \subset \omega_\beta(1) \subset \cdots \subset \omega_\beta(e_v) = \omega_\beta$$

such that $\mathscr{O}_F$ acts via $\tau \in \Sigma$ on $\omega_\tau := \omega_\beta(i)/\omega_\beta(i-1)$, when τ is of the form $\tau = \tau_\beta(i)$.

When $e_v = 1$, ω is locally free module of rank 1 over $\mathscr{O}_F \otimes_\mathbb{Z} \mathscr{O}_S$.

Automorphic bundle $\mathscr{A}_{(k,\ell)}$

- associated to $(k,\ell) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$, we have the automorphic line bundle

$$\mathscr{A}_{(k,\ell)} = \bigotimes_{\tau \in \Sigma} \omega_{\tau}^{k_\tau} \otimes \delta_{\tau}^{\ell_\tau}$$

where ω_τ is a locally-free-of-rank-1-over-$\mathscr{O}_{Y_\Gamma}$ piece of ω on which $\mathscr{O}_F$ acts via τ, and where

$$\delta = \bigwedge^{2}_{\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}} R^1 s_* \Omega^{\bullet}_{A/Y_\Gamma}$$

is free of rank 1 over $\mathscr{O}_F \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}$ and δ_τ is defined similarly.

Mod p HMFs

The space of mod p Hilbert modular forms of weight (k, ℓ) and of level Γ is defined to be

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)})$$

where $\overline{Y}_\Gamma = Y_\Gamma \times \overline{\mathbb{F}}_p$.

$F \neq \mathbb{Q}$

When $F \neq \mathbb{Q}$,

- every Hilbert modular form of weight (k, ℓ) over $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ has its weight paritious, i.e. $k_\tau + 2\ell_\tau$ is independent of τ.

  There are a lot more mod p Hilbert modular forms that are not in the image of

$$H^0(Y_\Gamma, \mathscr{A}_{(k,\ell)}) \to H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)}).$$

- In stark contract to the case $F = \mathbb{Q}$, there are lots of mod p Hilbert modular forms of '(partially) negative weights'.

Example ($\ell = 0$)

Emerton-Reduzzi-Xiao's partial Hasse invariants H_τ: the Verschibung $V = (\mathrm{Fr}_{A^\vee})^\vee : A^{(p)} \to A$ gives rise to

$$V^* : \overline{\omega}_\beta \to (\mathrm{Fr}^*\overline{\omega})_\beta = \overline{\omega}^p_{\phi^{-1}\circ\beta}$$

which breaks up into maps on $\overline{\omega}_{\tau_\beta(i)} = \overline{\omega}_\beta(i)/\overline{\omega}_\beta(i-1)$

$$\begin{tikzcd} \cdots \arrow[r] & \overline{\omega}_{\tau_\beta(e_v)} \arrow[r] \arrow[rrrr, bend left, "V^*"] & \overline{\omega}_{\tau_\beta(e_v-1)} \arrow[r] & \cdots \arrow[r] & \overline{\omega}_{\tau_\beta(1)} \arrow[r] & \overline{\omega}^p_{\tau_{\phi^{-1}\circ\beta}(e_v)} \arrow[r] & \cdots \end{tikzcd}$$

H_τ of weight

$$h_\tau = \begin{cases} 1\Phi^{-1}\tau + (-1)\tau & \text{if } \tau = \tau_\beta(i) \text{ for } e_v \geq i \geq 2, \\ p\Phi^{-1}\tau + (-1)\tau & \text{if } \tau = \tau_\beta(1) \end{cases}$$

Automorphic Galois representations

Theorem (DS)

Let f be an element $H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,\ell)})$ and S be a finite set of finite places in F, containing all v dividing p and all v such that $\mathrm{GL}_2(\mathscr{O}_{F_v}) \not\subset \Gamma$.
Suppose that

$$T_v f = \alpha_v f$$

and

$$S_v f = \beta_v f$$

for all v not in S. Then there exists a continuous representation

$$\overline{\rho}_f : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is unramified outside S and the characteristic polynomial in X of $\overline{\rho}_f(\mathrm{Frob}_v)$ is

$$X^2 - \alpha_v X + \beta_v \mathrm{N}_{F/\mathbb{Q}}(v).$$

Remark

The novelty of our theorem is that (k, ℓ) does not have to satisfy the parity condition ($k_\tau + 2\ell_\tau$ is independent of τ in Σ). The parity case is known by Emerton-Reduzzi-Xiao ($e_v = 1$), RX ($e_v \geq 1$), and Goldring-Koskivirta (general Shimura varieties).

Conjecture (Folklore)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be totally odd, continuous and irreducible. Then $\overline{\rho}$ is modular in the sense above.

Idea of our proof for the theorem when $e_v = 1$

Recall that not every $\overline{\rho}$ arises as the reduction of a characteristic zero eigenform of level prime to p.

How do we deal with HMFs of non-paritous weight? We lift mod p HMFs of parallel weight but of level $\Gamma \cap \Gamma_1(p)$.

The idea then is to establish congruences, i.e., find an eigenform of parallel weight $N + 2$ (to be specified) and level $\Gamma \cap \Gamma_1(p)$ which is congruent mod p (and Hasse) to f and which can be lifted to a characteristic zero eigenform when N is sufficiently large (the ampleness of a line bundle over X_Γ).

Preliminary reduction steps

• Γ may be taken to be a sufficiently small principal congruence subgroup of level prime to p,

• by twisting if necessary, WLOG $\ell_\tau = -1$ for every τ in Σ,

• (can always) find $N \in \mathbb{Z}$ and $r \in \mathbb{Z}^\Sigma$ such that $0 \leq r_\tau \leq p-1$ but not all r_τ are simultaneously $p-1$ such that $k - (N+2-r)$ is a linear combination of h_τ's. Multiplying H_τ's defines an injection of global sections $\rightsquigarrow k = N+2-r$.

[• the Katz-Mazur-Pappas covering

$$\pi : Y_{\Gamma \cap \Gamma_1(p)} \to Y_\Gamma$$

with extension

$$\pi : X_{\Gamma \cap \Gamma_1(p)} \to X_\Gamma$$

to minimal compactifications]

- observe that there is a Hecke equivariant injection

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,-1)}) \hookrightarrow H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})$$

with its image contained in

$$\begin{aligned} & H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota_* K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \\ \subset \quad & H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota}_* \overline{K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \\ = \quad & H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)}) \end{aligned}$$

where

- $\iota : Y_{\Gamma\cap\Gamma_1(p)} \hookrightarrow X_{\Gamma\cap\Gamma_1(p)}$,
- K denotes the dualising sheaf over the Cohen-Macaulay scheme $Y_{\Gamma\cap\Gamma_1(p)}$,
- and $\omega = \omega_{X_{\Gamma\cap\Gamma_1(p)}}$ here denotes the push-forward by ι of the analogue of $\mathscr{A}_{(1,0)}$ over $Y_{\Gamma\cap\Gamma_1(p)}$ (it is also the pull-back by π of the ample line bundle $\omega = \omega_{X_\Gamma}$ over X_Γ).

•

Theorem (DKS)

$$R^r\pi_*K = 0$$

for $r > 0$.

Proved by carefully describing fibres of the degeneracy map

$$\begin{array}{ccc} \overline{Y}_{\Gamma\cap\Gamma_0(p)} & \longrightarrow & \overline{Y}_{\Gamma} \\ \cup & & \cup \\ \text{(max GK-strata)} & \longrightarrow & \text{(EO/Hasse strata)} \end{array}$$

- the DKS theorem implies

$$R^1\pi_*(\iota_* K) = 0,$$

hence

$$\pi_*(\iota_* K) \to \pi_*(\overline{\iota_* K})$$

is surjective and, combined with the ampleness of ω, it follows that

$$\begin{array}{ccc} H^0(X_\Gamma, \pi_*\iota_* K \otimes_{\mathscr{O}_{X_\Gamma}} \omega^N) & \longrightarrow & H^0(X_\Gamma, \pi_*(\overline{\iota_* K}) \otimes_{\mathscr{O}_{X_\Gamma}} \overline{\omega}^N) \\ \| & & \| \\ H^0(X_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)}) & & H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota_* K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \end{array}$$

is surjective when N is sufficiently large, because

$$H^1(X_\Gamma, \pi_*\iota_* K \otimes_{\mathscr{O}_{X_\Gamma}} \omega^N) = 0.$$

To recap, the image of the Hecke equivariant injection

$$H^0(\overline{Y}_\Gamma, \mathscr{A}_{(k,-1)}) \hookrightarrow H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})$$

$(k = 2 + N - r)$ is contained in

$$H^0(\overline{X}_{\Gamma\cap\Gamma_1(p)}, \overline{\iota_* K} \otimes_{\mathscr{O}_{\overline{X}_{\Gamma\cap\Gamma_1(p)}}} \overline{\omega}^N) \twoheadleftarrow H^0(X_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)}).$$

To finish off, we make appeal to:

• there exists a Galois representation associated to an eigenform in $H^0(X_{\Gamma\cap\Gamma_1(p)}, \mathscr{A}_{(N+2,-1)})[1/p]$ by work of Carayol, Taylor,....

Q.E.D.

Remark

The $e_v > 1$ case can be dealt with similarly– to build an ample line bundle over X_Γ, we incorporate RX's observation about a line bundle relative ample over Y_Γ^{DP}.

In preparation of the DS conjecture

Definition (the Diamond-Kassaei minimal cone)

Let $\Xi^{\mathrm{DK}} \subset \mathbb{Z}^{\Sigma}$ be the set of $k = \sum_{\tau} k_\tau \tau$ such that

- $pk_\tau \geq k_{\Phi^{-1}\tau}$ if τ is of the form $\tau_\beta(1)$,
- $k_\tau \geq k_{\Phi^{-1}\tau}$ if τ is of the form $\tau_\beta(i)$ for $2 \leq i \leq e_v$.

And let Ξ be the subset of $k \in \Xi^{\mathrm{DK}}$ such that $k_\tau \geq 1$ for every τ in Σ.

Definition

$$k \succeq k'$$

if $k - k'$ is a non-negative integer linear combination of the weights h_τ of the partial Hasse invariants.

Conjecture

$\boxed{\text{Conjecture}}$ (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \succeq k(\overline{\rho}, \ell)$
- if $k \in \Xi$, then $k \succeq k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}_v = \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

Remark

Assuming $\overline{\rho}$ is (geometrically) modular, the existence of $k(\overline{\rho}, \ell)$ is suggested by DK– the weight filtration $w(f)$ of a mod p HMF f lies in Ξ^{DK}.

The first condition: ‘minimality’

Assuming the existence of $k(\overline{\rho}, \ell) \in \Xi$ satisfying the first condition, one sees that

- the conjecture implies the folklore conjecture earlier,
- the conjecture (i.e. the second condition) boils down to

$\boxed{\text{Conjecture } *}$ (DS)

Suppose that $\overline{\rho}$ is irreducible and modular. If $k \in \Xi$, then $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ).

The second condition: p-adic Hodge-theory

Assuming the existence of $k(\overline{\rho}, \ell) \in \Xi$ satisfying the second condition, $\boxed{\text{Conjecture}}$ follows from $\boxed{\text{Conjecture } *}$ if we know

$\boxed{\text{Conjecture } **}$ (DS)

Suppose that $\overline{\rho}$ is irreducible and $\overline{\rho} \simeq \overline{\rho}_f$ for some f. Then $w(f) \in \Xi$.

The second condition is suggested by the Breuil-Mézard conjecture and modular representation theory of $\mathrm{GL}_2(\mathbb{F}_v)$– it is somehow the underlying theme of DKS.

The qualification '$k \in \Xi$' in the second condition is needed– when $k \notin \Xi$, the condition $k \succeq k(\overline{\rho}, \ell)$ does not imply that $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ).

Example when $F = \mathbb{Q}$ and $\ell = 0$

$$\Xi^{\mathrm{DK}} = \{k \geq 0\},$$

$$\Xi = \{k \geq 1\},$$

$$k \succeq k' \text{ if } k - k' = (p-1)n \geq 0.$$

There exists $k(\overline{\rho}) \geq 1$ such that the following are equivalent:

- $\overline{\rho}$ is modular of weight k,
- $k \succeq k(\overline{\rho})$,
- $\overline{\rho}_p$ has a crystalline lift of weight $(k, 0)$,

for every $k \geq 1$.

$\rightsquigarrow$ $k(\overline{\rho})$ is the smallest possible weight for which $\overline{\rho}$ is modular.

How is our conjecture related to the Buzzard-Diamond-Jarvis conjecture?

Algebraic modularity (BDJ) for pedants

Let V be a finite-dimensional $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{F}_v)$. We say that $\overline{\rho}$ is algebraic modular of weight V if there exist

- a quaternion algebra D over F split at p and ramified at all but one place in $\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$,
- a sufficiently small open compact subgroup $U \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ ($\rightsquigarrow$ the Shimura curve Y_U over F) such that
  - U is of level prime to p (i.e. $\mathrm{GL}_2(\mathscr{O}_{F_v}) \subset U$),
  - if $U^+ = \ker(U \to \mathrm{GL}_2(\mathbb{F}_v))$, then $Y_{U^+} \to Y_U$ is étale of degree equal to $|\mathrm{GL}_2(\mathbb{F}_v)|$

such that $\overline{\rho}$ is an $\overline{\mathbb{F}}_p[\mathrm{Gal}(\overline{F}/F)]$-subquotient of $H^1_{\text{ét}}(Y_U \times \overline{F}, \mathcal{V})(1)$.

Algebraic modular = Geometric modular?

Conjecture (DS)

Let $(k, \ell) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and $k_\tau \geq 2$ for every τ in Σ. If $\overline{\rho}$ is algebraic modular of weight (k, ℓ), i.e., of representation weight

$$V_{k,1-k-\ell} = \bigotimes_\tau \mathrm{Sym}^{k_\tau - 2} \mathrm{det}^{1-k_\tau-\ell_\tau} (V_{\mathrm{st}} \otimes_\tau \overline{\mathbb{F}}_p),$$

(where V_{st} is the standard representation of $\mathrm{GL}_2(\mathbb{F}_v)$ on two copies of $\mathbb{F}_v$) then $\overline{\rho}$ is modular of weight (k, ℓ).

If furthermore $k \in \Xi$, the converse holds. This is false if $k \notin \Xi$!

Remarks

We know that if $\overline{\rho}$ is algebraic modular of paritious weight (k, ℓ), then $\overline{\rho}$ is modular of weight (k, ℓ).

By our construction of modular Galois representations, if $\overline{\rho}$ is modular of some weight, $\overline{\rho}$ is algebraic modular of some weight.

Conjecture (Recap)

$\boxed{\text{Conjecture}}$ (DS)

Let

$$\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p),$$

totally odd, continuous, irreducible.

Fix ℓ in $\mathbb{Z}^{\Sigma}$. Then there exists $k(\overline{\rho}, \ell)$ lying in Ξ satisfying the following conditions:

- $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $k \succeq k(\overline{\rho}, \ell)$
- if $k \in \Xi$, then $k \succeq k(\overline{\rho}, \ell)$ if and only if $\overline{\rho}_v = \overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_v)}$ has a crystalline lift of weight (k, ℓ), i.e. of Hodge-Tate weight $(k + \ell - 1, \ell)$.

(recap)

Assuming 'minimal weight' $k(\overline{\rho}, \ell) \in \Xi$ exists (that satisfies the first condition), the second condition boils down to:

$\boxed{\text{Conjecture } *}$ (DS)

Suppose that $\overline{\rho}$ is irreducible and modular. If $k \in \Xi$, then $\overline{\rho}$ is modular of weight (k, ℓ) if and only if $\overline{\rho}_v$ has crystalline lift of weight (k, ℓ).

Example when $[F:\mathbb{Q}] = 2$, $e_v = 0$ and $\ell = 0$

Suppose $[F:\mathbb{Q}] = 2$ and $e_v = 0$. Hence $\Sigma = \Sigma^r$. Fixing τ in Σ,

$$\Sigma = \{\tau, \phi \circ \tau = \phi^{-1} \circ \tau\}$$

and we will write weights labelled by τ on the left of every pair.

We furthermore assume $\ell = (0,0)$.

Theorem (DS)

Let $2 < r \leq p$. Suppose that $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular.

If $\overline{\rho}_v$ has crystalline lift of weight

$$((r,1),(0,0))$$

(=HT weight $((r-1,0),(0,0))$) then $\overline{\rho}$ is (geometric) modular of weight

$$((r,1),(0,0)).$$

Proof (Sketch)

For brevity, we furthermore assume that $\overline{\rho}$ is of Taylor-Wiles type–the exceptional case can be dealt with by an ad hoc argument.

Step 1

Given that $\overline{\rho}_v$ has crystalline lift of weight

$$((r,1),(0,0)),$$

we deduce from p-adic Hodge theory that $\overline{\rho}_v$ also has crystalline lifts of weight

$$(k,\ell) = ((r-1,p+1),(0,0))$$

and

$$(k',\ell') = ((r+1,p+1),(-1,0)).$$

Step 2: algebraic companion forms

By work of Gee and his collaborators on the BDJ conjecture that $\overline{\rho}$ is algebraic modular of weight (k, ℓ) (=weight $V_{k,1-k-\ell}$) and (k', ℓ').

Step 3: geometric companion forms

Since r and 1 are paritious, $\overline{\rho}$ is geometric modular of weight (k, ℓ) and (k', ℓ').

Step 4: combinatorics

Let f (resp. f') be a geometric HMF of weight (k, ℓ) (resp. (k', ℓ')) such that $\overline{\rho}_f \simeq \overline{\rho}$ (resp. $\overline{\rho} \simeq \overline{\rho}_{f'}$). One observes

- $\theta_\tau(f)$ is of weight

  $$((r-1, p+1), (0, 0)) + ((1, p), (-1, 0)) = ((r, 2p+1), (-1, 0)),$$

- $f' H_\tau$ is an eigenform of weight

  $$((r+1, p+1), (-1, 0)) + ((-1, p), (0, 0)) = ((r, 2p+1), (-1, 0)),$$

- $\theta_\tau(f) = f' H_\tau$ (by replacing them by forms of higher level if necessary– so that q-expansion coefficients at 'bad primes' are 0)

θ-operators/cycles

Theorem (AG)

For any HMF f of weight (k, ℓ), the image $\theta_\tau(f)$ is divisible by H_τ if and only if f is divisible by H_τ or p divides k_τ.

Step 5: a mod p HMF of partial weight one

Deduce from the theorem that f is divisible by H_τ. The HMF f/H_τ of weight

$$((r-1,p+1),(0,0))-((-1,p),(0,0))=((r,1),(0,0))$$

is what we are looking for.

Remarks

The argument is reversible except that we do not know the 'algebraic modular = geometric modular' conjecture (in particular we do not know 'geometric modular $\Rightarrow$ algebraic modular' even in the paritious case).

When $[F:\mathbb{Q}] = 2 = e_v$ and $\Sigma = \{\tau(1), \tau(2)\}$, we can prove: if $\overline{\rho}_v$ has crystalline lift of weight $((1,r),(0,0))$, then $\overline{\rho}$ is modular of weight $((1,r),(0,0))$.

Final remark

We have seen the interplay between algebraic weights and geometric weights:

$$V_{k,1-k-\ell} \leftrightsquigarrow \mathscr{A}_{(k,\ell)}.$$

In my work with FD and P. Kassaei, we make intrinsic connections between mod p representations of $\mathrm{GL}_2(\mathbb{F}_v)$ and mod p geometry of the Shimura variety for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$:

- the JH factors in $\mathrm{Ind}_{B(\mathbb{F}_v)}^{\mathrm{GL}_2(\mathbb{F}_v)}\chi$, where χ is a character $\mathbb{F}_v^\times \to \overline{\mathbb{F}}_p^\times$
- the graded pieces of a filtration on the χ-isotypic component of the space $H^0(\overline{Y}_{\Gamma\cap\Gamma_1(p)}, K)$ of mod p HMFs of parallel weight 2 and level $\Gamma \cap \Gamma_1(p)$

Thank you very much for listening.